Points of Inflexion

Starter

1. **(Review of last lesson)** Evaluate $\int_1^2 \frac{3x+5}{x(x+10)} dx$ giving your answer to 3 s.f.

2. **(Review of AS material)** Consider the curve $y = x^3$. Find
   (a) the stationary point
   (b) the value of the second derivative at the stationary point
   (c) the signs of the gradient to the left and right of the stationary point.

3. **(Review of AS material)** Repeat question 2 for the curve $y = x^4$.

Notes

The point $(0, 0)$ on the curve $y = x^3$ is called a ***point of inflexion*** (or AmE: inflection). It is a point where the curve ***changes from concave-up to concave-down, or vice versa*** (as in the case of $y = x^3$.

Finding points of inflexion

From the starter, we found that $\frac{d^2y}{dx^2} = 0$ at the point of inflexion on the curve $y = x^3$ but that the curve $y = x^4$ also had $\frac{d^2y}{dx^2} = 0$ at the origin but this point was not a point of inflexion.

We say that $\frac{d^2y}{dx^2} = 0$ is a ***necessary but not sufficient*** condition for the existence of a point of inflexion.

We also need to check the ***gradient either side of a suspected point of inflexion***: if the gradient either side is ***both positive*** or ***both negative***, there is a point of inflexion.

Success Criteria - finding a point of inflexion

1. Put $\frac{d^2y}{dx^2} = 0$ and solve — this gives s a possible point of inflexion
2. Choose $x-$values either side of the possible point of inflexion and find the gradient at these points (substitute into $\frac{dy}{dx}$).
   Sign of the gradient is the same either side — point of inflexion
   Sign of the gradient is different either side — maximum or minimum

Type	Gradient change
Maximum	+ve/—ve
Minimum	—ve/+ve
Increasing point of inflection	+ve/+ve
Decreasing point of inflection	—ve/—ve

E.g. 1 Find and classify the stationary point(s) and point(s) of inflexion of the curve $f(x) = x^4 - 4x^3$. Hence sketch the graph.

Working: $f'(x) = 4x^3 - 12x^2$

$f'(x) = 0$ when $x = 3$ and $x = 0$ ***these are stationary points***

$f''(x) = 12x^2 - 24x$

$f''(3) > 0$ so when $x = 3$ there is a minimum

When $x = 3, y = -27$ so $(3, -27)$ is a minimum

$f''(0) = 0$ so $x = 0$ is a ***possible*** point of inflexion

Solving $f''(x) = 0$ gives $x = 0$ and $x = 2$ – possible points of inflexion

Now choose $x-$values either side of $x = 0$ and $x = 2$ to check gradient

$x = 0$: $f'(-0.1) < 0$

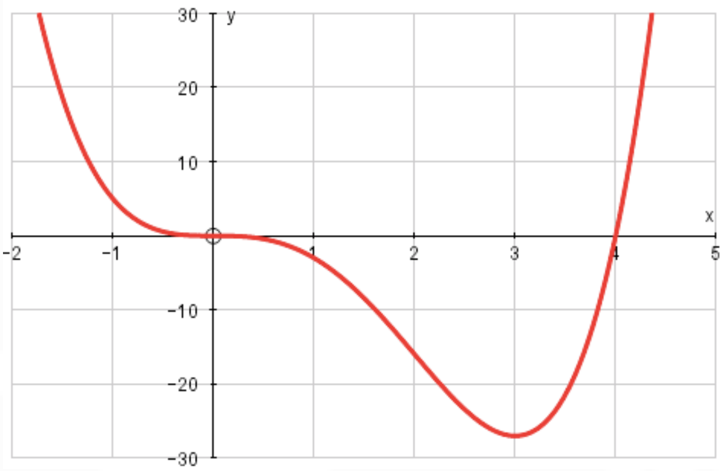
$f'(0.1) < 0$

When $x = 0, y = 0$ so $(0, 0)$ is a point of inflection

$x = 2$: $f'(1.9) < 0$

$f'(2.1) < 0$

When $x = 2, y = -16$ so $(2, -16)$ is a point of inflection

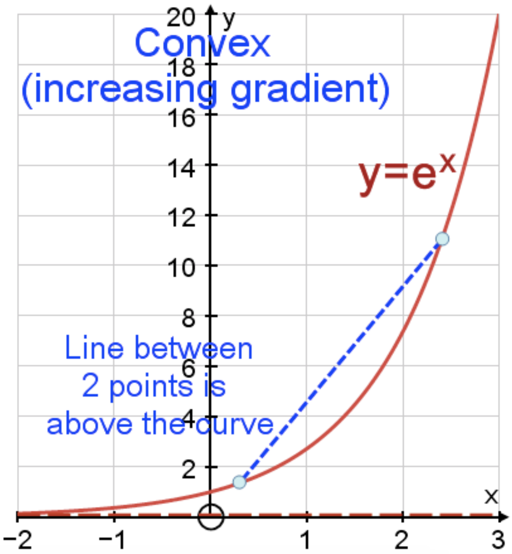


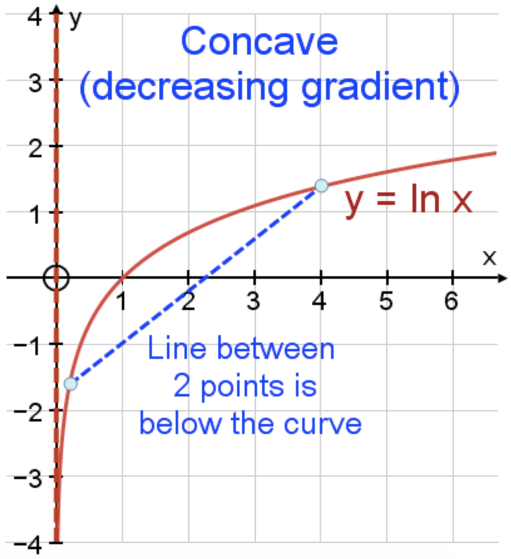
E.g. 2 Find the point(s) of inflexion of the graph of $y = 2x^4 + 4x^3 - 72x^2$.

Convex and concave curves

Convex ⇒ concave-up $\frac{d^2y}{dx^2} > 0$ Concave ⇒ concave-down $\frac{d^2y}{dx^2} < 0$

("vex" think e^x)





N.B. Most curves are not entirely convex or concave but can be divided into convex and concave sections

	Convex	Concave
Gradient	Increasing	Decreasing
Second derivative	> 0	< 0
Examples	$y = x^2$, $y = e^x$	$y = -x^2$, $y = \ln x$

E.g. 3 Find the range of $x-$values for which the curve $y = x^3 + x^2 - x$ is convex.

Working: $\frac{dy}{dx} = 3x^2 + 2x - 1$

$\frac{d^2y}{dx^2} = 6x + 2$

The curve is convex when $\frac{d^2y}{dx^2} > 0$ ⇒ $6x + 2 > 0$

$x > -\frac{1}{3}$

E.g. 4 Find the values for which the curve $f(x) = \frac{1}{3}x^3 - x^2 - 15x$ is concave:

Video: Points of inflexion
Video: Convex and concave curves

Solutions to Starter and E.g.s

Exercise
p252 12A Qu 4-10

Summary
Finding a point of inflexion:

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    Sign of the gradient is the same either side — point of inflexion
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Convex ⇒ concave-up $\quad \dfrac{d^2y}{dx^2} > 0$

Concave ⇒ concave-down $\quad \dfrac{d^2y}{dx^2} < 0$