

Points of Inflexion

Starter

- (Review of last lesson)** Evaluate $\int_1^2 \frac{3x + 5}{x(x + 10)} dx$ giving your answer to 3 s.f.
- (Review of AS material)** Consider the curve $y = x^3$. Find
 - the stationary point
 - the value of the second derivative at the stationary point
 - the sign of the gradient to the left and right of the stationary point.
- (Review of AS material)** Repeat question 2 for the curve $y = x^4$.

Notes

The point $(0, 0)$ on the curve $y = x^3$ is called a **point of inflexion** (or AmE: inflection).

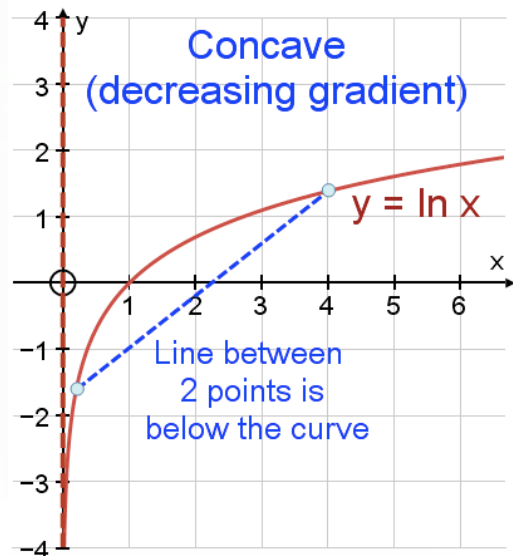
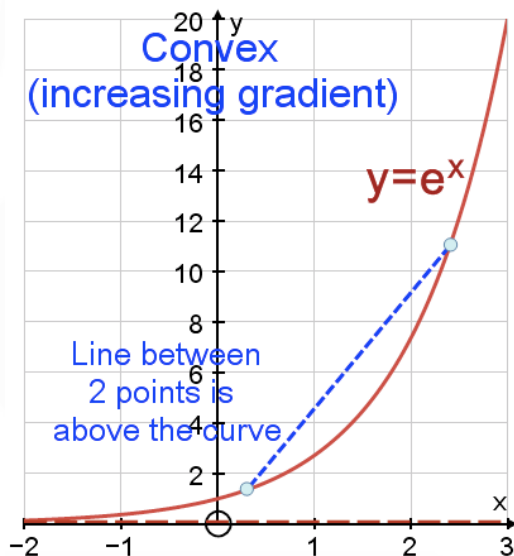
Definition – point of inflexion

A point of inflexion is where the curve **changes from convex** (concave-up) **to concave** (concave-down), **or vice versa** (as in the case of $y = x^3$).

Convex and concave curves

Convex \Rightarrow concave-up $\frac{d^2y}{dx^2} > 0$
 (“vex” think e^x)

Concave \Rightarrow concave-down $\frac{d^2y}{dx^2} < 0$



	Convex	Concave
Gradient	Increasing	Decreasing
Second derivative	> 0	< 0
Examples	$y = x^2, y = e^x$	$y = -x^2, y = \ln x$

N.B. Most curves are not entirely convex or concave but can be divided into convex and concave sections.

E.g. 1 Find the range of x -values for which the curve $y = x^3 + x^2 - x$ is convex. Give your answer in set notation.

Working: $\frac{dy}{dx} = 3x^2 + 2x - 1$
 $\frac{d^2y}{dx^2} = 6x + 2$

The curve is convex when $\frac{d^2y}{dx^2} > 0 \Rightarrow 6x + 2 > 0$

$$x > -\frac{1}{3}$$

In set notation: $\left\{ x : x > -\frac{1}{3} \right\}$

E.g. 2 Find the values for which the curve $f(x) = x^4 + 4x^3 - 18x^2 + 7$ is concave. Give your answer in set notation.

Finding points of inflexion

From the starter, we found that $\frac{d^2y}{dx^2} = 0$ at the point of inflexion on the curve $y = x^3$. However, the curve $y = x^4$ also had $\frac{d^2y}{dx^2} = 0$ at the origin but this was not a point of inflexion.

We say that $\frac{d^2y}{dx^2} = 0$ is a **necessary but not sufficient** condition for the existence of a point of inflexion.

To decide whether it is a point of inflexion or not, return to the definition:

A point of inflexion is where the curve **changes from convex** (concave-up) **to concave** (concave-down), **or vice versa**.

Therefore, check the value of the **second derivative either side of a suspected point of inflexion**: if the second derivative **changes sign**, there is a point of inflexion.

Success Criteria - finding a point of inflexion

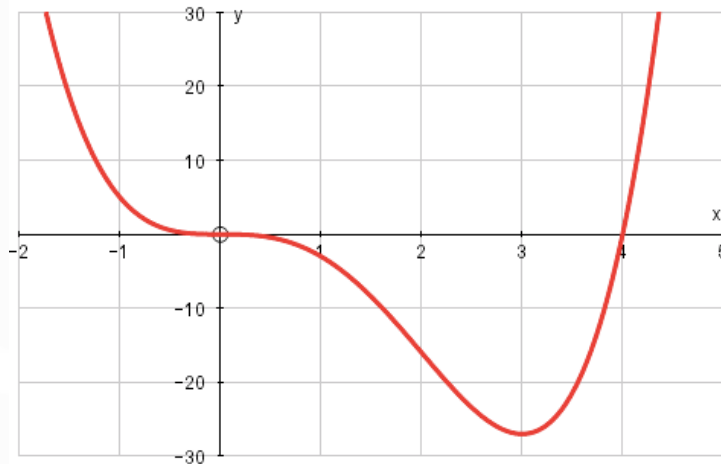
1. Put $\frac{d^2y}{dx^2} = 0$ and solve — this gives a possible point of inflexion
2. Choose x -values either side of the possible point of inflexion and find the value of the second derivative at these points (substitute into $\frac{d^2y}{dx^2}$).

Change of sign \Rightarrow point of inflexion

No change of sign \Rightarrow no point of inflexion

E.g. 3 Find and classify the stationary point(s) and point(s) of inflexion of the curve $f(x) = x^4 - 4x^3$. Hence sketch the graph.

Working: $f'(x) = 4x^3 - 12x^2$
 $f'(x) = 0$ when $x = 3$ and $x = 0$ *these are stationary points*
 $f''(x) = 12x^2 - 24x$
 $f''(3) > 0$ so when $x = 3$ there is a minimum
 When $x = 3$, $y = -27$ so $(3, -27)$ is a minimum
 $f''(0) = 0$ so $x = 0$ is a **possible** point of inflexion
 Solving $f''(x) = 0$ gives $x = 0$ and $x = 2$ – possible points of inflexion
 Now choose x -values either side of $x = 0$ and $x = 2$ to check gradient
 $x = 0$: $f'(-0.1) < 0$
 $f'(0.1) < 0$
 When $x = 0$, $y = 0$ so $(0, 0)$ is a point of inflexion
 $x = 2$: $f'(1.9) < 0$
 $f'(2.1) < 0$
 When $x = 2$, $y = -16$ so $(2, -16)$ is a point of inflexion



E.g. 4 Find the point(s) of inflexion of the graph of $y = 2x^4 + 4x^3 - 72x^2$.

[Video: Points of inflexion](#)

[Video: Convex and concave curves](#)

[Solutions to Starter and E.g.s](#)

Exercise

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Summary

Convex \Rightarrow concave-up $\frac{d^2y}{dx^2} > 0$

Concave \Rightarrow concave-down $\frac{d^2y}{dx^2} < 0$

Definition — point of inflexion:

A point of inflexion is where the curve **changes from convex** (concave-up) **to concave** (concave-down), **or vice versa** (as in the case of $y = x^3$).

Finding a point of inflexion:

1. Put $\frac{d^2y}{dx^2} = 0$ and solve — this gives a possible point of inflexion
2. Choose x -values either side of the possible point of inflexion and find the gradient at these points (substitute into $\frac{dy}{dx}$).

Sign of the gradient is the same either side — point of inflexion

Sign of the gradient is different either side — maximum or minimum

