Product Rule

Starter

1. **(Review of last lesson)**
   Differentiate these functions: (a) $f(x) = \cos^8(6x+5)$ (b) $y = e^{\cos(4x-5)}$

2. Factorise $3x^2(3x-2)^7 + 21x^3(3x-2)^6$.

3. Show that $2x\sqrt{6x-1} + \dfrac{3x^2}{\sqrt{6x-1}}$ can factorise to $\dfrac{x(15x-2)}{\sqrt{6x-1}}$.

Notes

Remember: from first principles, the derivative of $y = f(x)$ is given by:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{x + \delta x - x} \qquad \text{or} \qquad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y + \delta y - y}{x + \delta x - x}$$

The ***product rule*** is used to differentiate functions that are comprised of two separate functions multiplied together. For example, $y = x^2(5x-1)^7$ or $y = x\sqrt{x-7}$.

Derivation of the product rule

Let $y = uv$ where u and v are functions of x i.e. $u = u(x)$ and $v = v(x)$.
Let x increase by a small amount δx and let the corresponding increases in y, u and v be δy, δu and δv respectively.

So $y + \delta y = (u + \delta u)(v + \delta v) = uv + v\delta u + u\delta v + \delta u \delta v$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y + \delta y - y}{x + \delta x - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{uv + v\delta u + u\delta v + \delta u \delta v - uv}{\delta x} \qquad \textit{(remember } y = uv\textit{)}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{v\delta u + u\delta v + \delta u \delta v}{\delta x} \qquad \textit{(simplifying)}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(v\frac{\delta u}{\delta x} + u\frac{\delta u}{\delta x} + \delta v\frac{\delta u}{\delta x} \right) \qquad \textit{(separating into individual terms)}$$

Remember from AS material, as $\delta x \to 0$, $\quad \delta u \to 0$ and $\delta v \to 0$

so $\quad \dfrac{\delta u}{\delta x} \to \dfrac{du}{dx}$ and $\dfrac{\delta v}{\delta x} \to \dfrac{dv}{dx}$

Also $\delta v \dfrac{\delta u}{\delta x} \to \delta v \dfrac{du}{dx} \to 0$ since $\delta v \to 0$

So $\quad \dfrac{dy}{dx} = \dfrac{du}{dx}v + \dfrac{dv}{dx}u \qquad$ **this is the product rule**

Or $\quad \dfrac{dy}{dx} = u'v + v'u$

Poetry for the product rule

Differentiate first term,	u'
Times by second,	$u'v$
Add,	$u'v+$
Differentiate second term,	$u'v + v'$
Times by first.	$u'v + v'u$

(Romeo and Juliet, Act II Scene 2, “Balcony Scene”)

N.B. Often the most difficult part of using the product rule is the factorising at the end
Questions often require you to be able to use the chain rule

E.g. Find $\dfrac{dy}{dx}$ when $y = e^x \sin x$.

Working:

$u = e^x \qquad u' = e^x$

$v = \sin x \qquad v' = \cos x$

$$\frac{dy}{dx} = e^x \times \sin x + \cos x \times e^x$$
$$= e^x(\sin x + \cos x)$$

E.g. 1 Differentiate $y = x^2(3x+1)^5$ with respect to x.

Working: Using the poetry, you can go straight to line** below.
However, many students find this structure helpful:

$u = x^2 \qquad u' = 2x$

$v = (3x+1)^5 \qquad v' = 15(3x+1)^4$

$$\frac{dy}{dx} = 2x \times (3x+1)^5 + 15(3x+1)^4 \times x^2 \qquad **$$
$$\frac{dy}{dx} = x(3x+1)^4\left[2(3x+1) + 15x\right]$$
$$\frac{dy}{dx} = = x(21x+2)(3x+1)^4$$

E.g. 2 Differentiate: (a) $f(x) = 3x^4 \cos x$. (b) $y = 6x\sqrt{4x-1}$

E.g. 3 Find the stationary point(s) for the curve $y = xe^{x-x^2}$.

Video: Product rule

Solutions to Starter and E.g.s

Exercise
p205 10B Qu 1i, 2i, 3-10, (FM — 11)

Summary

Product rule: $\dfrac{dy}{dx} = \dfrac{du}{dx}v + \dfrac{dv}{dx}u$ or $\dfrac{dy}{dx} = u'v + v'u$

Poetry: Differentiate first term,
Times by second,
Add,
Differentiate second term,
Times by first.