Proof by contradiction

Starter

1. *(Review of last lesson)* Prove that the square of an odd number is always odd.

Notes

Proof by contradiction assumes that the opposite is true and then a series of logical arguments are followed which lead to an incorrect conclusion i.e. a contradiction of the original assumption.

The working includes four parts:

- **Opposite:** Assume the opposite is true.
- **Working:** Use algebraic arguments based on the assumption that the opposite is true.
- **Contradiction:** Reach the point where the argument contradicts the assumption.
- **Conclusion:** Conclude that the original statement in the question is true.

E.g. 1 Prove by contradiction that $\tan x - \sin x > 0$ for $0° < x < 90°$.

Working:

Opposite: Assume that $\tan x - \sin x \leq 0$ for $0° < x < 90°$.

Working:

$\Rightarrow \quad \dfrac{\sin x}{\cos x} - \sin x \leq 0$

$\Rightarrow \quad \sin x\left(\dfrac{1}{\cos x} - 1\right) \leq 0$

$\Rightarrow$ For $0° < x < 90°$, $\sin x > 0$

$\Rightarrow \quad \dfrac{1}{\cos x} - 1 \leq 0$

$\Rightarrow \quad \dfrac{1}{\cos x} \leq 1$

Since for $0° < x < 90°$, $\cos x > 0$:

$\Rightarrow \quad \cos x \geq 1$

Contradiction: But this is a contradiction since for $0° < x < 90°$, $0 < \cos x < 1$.

Conclusion: Therefore, $\tan x - \sin x > 0$ for $0° < x < 90°$.

Infinite proofs

E.g. 2 Prove that there are an infinite number of even numbers.

E.g. 3 Prove that there are an infinite number of prime numbers.

Working: **Opposite:** Assume that there are a finite number of prime numbers.

Working: $\Rightarrow$ All the prime numbers can be written as $p_1, p_2, p_3, \ldots, p_n$ where p_n is the largest prime number.

Let P be the product of all the prime numbers.

$\Rightarrow \quad P = p_1 \times p_2 \times p_3 \times \ldots \times p_n$

$\Rightarrow \quad P + 1 = p_1 p_2 p_3 \ldots p_n + 1$

Divide $P+1$ by p_1: $\dfrac{P+1}{p_1} = \dfrac{p_1 p_2 p_3 \ldots p_n + 1}{p_1}$

$\Rightarrow \quad \dfrac{P+1}{p_1} = \dfrac{p_1 p_2 p_3 \ldots p_n}{p_1} + \dfrac{1}{p_1}$

$\Rightarrow \quad \dfrac{P+1}{p_1} = p_2 p_3 \ldots p_n + \dfrac{1}{p_1}$

$\Rightarrow \quad P+1$ gives a remainder of 1 when divided by p_1

$\Rightarrow \quad P+1$ is not divisible by p_1

Similarly, $P+1$ is not divisible by $p_2, p_3, \ldots, p_n$.

If $P+1$ is prime, then it is a new prime not on the list.

If $P+1$ is not prime, then it must be the product of primes not on the list.*

Contradiction: Hence, $p_1, p_2, p_3, \ldots, p_n$ is not a complete list of prime numbers.

Conclusion: Therefore, there are an infinite number of prime numbers.

*For example, imagine our list of primes is $2, 3, 5, 7, 11, 13$.
Then $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
i.e. $P+1$ is the product of new primes not on the list

Irrational numbers

Irrational number are non-repeating decimals and cannot be expressed as a fraction of the form $\dfrac{p}{q}$ where p and q are integers. The numbers π and e are irrational and later in the lesson, it will be proved that $\sqrt{2}$ is also irrational.

E.g. 4 Prove by contradiction that the product of a rational number with an irrational number is an irrational number.

Working: Let p be a rational number such that $p = \dfrac{a}{b}$, where a and $b \neq 0$ are integers and let q be an irrational number.

Assume the product of p and q is rational:

$\Rightarrow \quad pq = \dfrac{c}{d}$ where c and $d \neq 0$ are integers

$\Rightarrow \quad \dfrac{a}{b} \times q = \dfrac{c}{d}$ since $p = \dfrac{a}{b}$

$\Rightarrow \quad q = \dfrac{bc}{ad}$

But this is the form of a rational number, since bc and ad are integers, which is a contradiction since q is an irrational number.

Hence, the product of a rational number with an irrational number is an irrational number

Key step for proof of irrationality

For the proof that $\sqrt{2}$ is irrational, the key step is that if p^2 is divisible by 2, then p is also divisible by 2.

E.g. 5 (a) Use proof by contradiction to prove that if p^2 is divisible by 2, then p is also divisible by 2.

(b) Write down a similar proof for 3.

(c) Does the proof work for the number 4? Explain your answer.

(d) If p^2 is divisible by k, then p is also divisible by k. Conjecture which numbers this is not always true for.

Working: (a) Let p^2 is divisible by 2.
Assume p is not divisible by 2:
$\Rightarrow$ p is odd
$\Rightarrow$ p is of the form $2a+1$ where a is an integer
$\Rightarrow$ $p^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$
But this is the form of an odd number and contradicts that p^2 is divisible by 2.
Therefore, by proof by contradiction, if p^2 is divisible by 2, then p is also divisible by 2.

Key language — co-prime

Integers that are ***co-prime*** only have the number 1 as a common factor. Therefore, if a and b are co-prime, the fractions $\frac{a}{b}$ and $\frac{b}{a}$ cannot be cancelled.

Proof of irrationality

In the proof of irrationality, it can be assumed that if p^2 is divisible by k, then p is also divisible by k (so long as k is not a square number or not a multiple of a square number).

E.g. 6 Prove that $\sqrt{2}$ is an irrational number.

Working:

Opposite: Assume that $\sqrt{2}$ is a rational number:

Working:
$\Rightarrow$ $\sqrt{2} = \frac{p}{q}$ where p and q are co-prime integers.
$\Rightarrow$ $2 = \frac{p^2}{q^2}$
$\Rightarrow$ $p^2 = 2q^2$ so p^2 is a multiple of 2
$\Rightarrow$ p is a multiple of 2, so $p = 2a$ where a is an integer
$\Rightarrow$ $(2a)^2 = 2q^2$
$\Rightarrow$ $4a^2 = 2q^2$
$\Rightarrow$ $2a^2 = q^2$ so q^2 is a multiple of 2
$\Rightarrow$ q is a multiple of 2

Contradiction: But this is a contradiction because p and q are co-prime.

Conclusion: Therefore, $\sqrt{2}$ is an irrational number.

E.g. 7 Prove that $\sqrt{5}$ is irrational.

E.g. 8 Prove that $\log_3 5$ is irrational.

Working:

Opposite: Assume that $\log_3 5$ is rational:

Working: $\Rightarrow \quad \log_3 5 = \dfrac{p}{q}$ where p and q are co-prime integers.

$\Rightarrow \quad 5 = 3^{\frac{p}{q}}$

$\Rightarrow \quad 5^q = 3^p$

Contradiction: But this is a contradiction because the integer powers of 5 end in 5 and the integer powers of 3 end in $1, 3, 7$ or 9.

Conclusion: Therefore, $\log_3 5$ is irrational.

Video: Proof by contradiction

Solutions to Starter and E.g.s

Exercise

p4 1B Qu 1-9, (10-12 red)

Summary

In the proof of irrationality, it can be assumed that if p^2 ***is divisible by*** k, then p ***is also divisible by*** k (so long as k is not a square number).