Separable Differential Equations

Starter

1. (Review of last lesson) Find the particular solution of the differential equation $\frac{dy}{dx} = e^{2x} - \sin 3x$ given that $x = 0$ when $y = 2$.

Notes

So far we have solved differential equations of the form $\frac{dy}{dx} = f(x)$. They become more difficult when a function of y appears.

E.g. $\frac{dy}{dx} = xy$ $\quad\quad$ $\frac{dy}{dx} = e^x \sin y$

Such equations are of the form $\frac{dy}{dx} = f(x)g(y)$ i.e. the functions in x and y ***can be separated***.
To solve such equation, the variables are separated — the function in y goes with dy and the function in x goes with dx.

i.e. $\int \frac{1}{g(y)} dy = \int f(x) dx$ $\quad$ ***both sides are then integrated separately***

E.g. 1 Find the general solution to the differential equation $\frac{dy}{dx} = \frac{x}{y^2 - 2}$.

Working: $\frac{dy}{dx} = \frac{x}{y^2 - 2}$

Separate the variables: $\int (y^2 - 2) dy = \int x dx$

Integrate both sides: $\frac{1}{3}y^3 - 2y = \frac{1}{2}x^2 + c$

Multiply by 6: $2y^3 - 12y = 3x^2 + 6c$

Replace $6c$ by constant A: $2y^3 - 12y = 3x^2 + A$

N.B. Write the constant in the simplest form so A is better than $6c$.

When integrating both sides, we combine the two constants that should appear into one constant.

N.B. If $\frac{dy}{dx} = 3xy$ do not move the 3 so $\int \frac{1}{y} dy = \int 3x dx$

If $\frac{dy}{dx} = -y$ do not move the −ve sign $\int \frac{1}{y} dy = -\int dx$

E.g. 2 Which of the following can be solved using the method of separation of variables? Solve those to which the method applies.

(a) $\frac{dy}{dx} = 2x + xy$ $\quad\quad$ (b) $\frac{dy}{dx} = 3x - 2y$

(c) $\frac{dy}{dx} = xy + 6$ $\quad\quad$ (d) $3x + 2y\frac{dy}{dx} = 5$

E.g. 3 Find the particular solution of the differential equation $y^2\dfrac{dy}{dx} = x^2 + 1$ given that $y = 1$ when $x = 2$

Working: $y^2\dfrac{dy}{dx} = x^2 + 1 \quad \Rightarrow \quad \int y^2 dy = \int (x^2 + 1)dx$

Integrate both sides: $\dfrac{1}{3}y^3 = \dfrac{1}{3}x^3 + x + c$

Multiply by 3: $y^3 = x^3 + 3x + A$

when $x = 2, y = 1$: $1 = 2^3 + 3 \times 2 + A \quad \Rightarrow \quad A = -13$

The particular solution is $y^3 = x^3 + 3x - 13$

E.g. 4 Find the general solution of the differential equation $\dfrac{dy}{dx} = y - y\cos x$.

E.g. 5 Find the particular solution of the differential equation $\dfrac{dy}{dx} = \sin x \cos^2 y$ given that the curve passes through $\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

E.g. 6* If $e^{-x^2}\dfrac{dy}{dx} = x(y+2)^2$, find y in terms of x, given that $y = 0$ when $x = 0$.

Video: Separation of variables
Video: Separation of variables (ln)
Video: Separation of variables (exp/trig)

Solutions to Starter and E.g.s

Exercise
p288 13B Qu 1i, 2i, 3i, 4-7

Summary

$\dfrac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \int \dfrac{1}{g(y)}dy = \int f(x)dx$ ***both sides integrated separately***