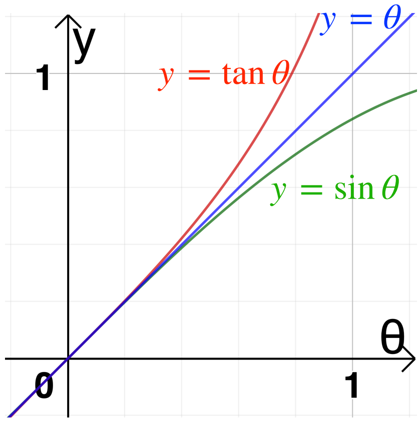
Small angle approximations

Starter

1. Prove the identity $\dfrac{\sin A}{\sin B} + \dfrac{\cos A}{\cos B} \equiv \dfrac{2\sin(A+B)}{\sin 2B}$.

Notes

Consider the graphs of $y = \theta$, $y = \tan\theta$ and $y = \sin\theta$, where θ is measured in radians.



As can be seen, the closer the angle is to 0^c, the closer the graphs are. That is for small angles:

$$\theta \approx \tan\theta \approx \sin\theta$$

From the graphs, the value of $\sin\theta$ is closer to θ for longer as θ increases.

This can be seen from the table below:

Angle in degrees, θ^o	**1°**	**5°**	**10°**	**20°**
Angle in radians, θ^c	0.017 453 3	0.087 266 5	0.174 532 9	0.349 065 9
sin θ	0.017 452 4	0.087 155 7	0.173 648 2	0.342 020 1
tan θ	0.017 455 1	0.087 488 7	0.176 327 0	0.363 970 2

The small angle approximations are reasonable up to about 0.35^c i.e. $20°$

Small angle approximation for $\cos\theta$

Since $\cos^2\theta = 1 - \sin^2\theta$: $\qquad \cos\theta \approx \sqrt{1-\theta^2} = (1-\theta^2)^{\frac{1}{2}}$.

E.g. 1 (a) Find the expansion of $\sqrt{1-\theta^2}$ up to and including the term in θ^6.

(b) When θ is small, powers above 2 can be ignored. Write down the approximation for $\cos\theta$.

Working: (a)

$$\sqrt{1-\theta^2} = (1-\theta^2)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-\theta^2) + \frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(-\theta^2)^2 + \frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{3!}(-\theta^2)^3 + \dots$$
$$= 1 - \frac{1}{2}\theta^2 - \frac{1}{8}\theta^4 - \frac{1}{16}\theta^6 + \dots$$

(b) The small angle approximation for $\cos\theta$ is $1 - \dfrac{1}{2}\theta^2$

The small angle approximations are:

$$\theta \approx \tan\theta \approx \sin\theta \qquad \text{and} \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2$$

N.B. $\sin 2\theta \approx 2\theta$

$\cos 2\theta \approx 1 - \frac{1}{2}(2\theta)^2 = 1 - 2\theta^2$

E.g. 2 Find approximations for the following when θ is small:

(a) $\frac{1-\cos\theta}{\theta^2}$ (b) $\frac{\theta\sin\theta}{1-\cos 2\theta}$ (c) $\frac{21+7\tan\theta-20\cos\theta}{1+\sin 2\theta}$

Working: (a) $\frac{1-\cos\theta}{\theta^2} \approx \frac{1-(1-\frac{1}{2}\theta^2)}{\theta^2} = \frac{\frac{1}{2}\theta^2}{\theta^2} = \frac{1}{2}$

E.g. 3 Find: (a) $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}$ (b) $\lim_{\theta\to 0}\frac{\tan 7\theta}{\sin\theta}$ (c) $\lim_{\theta\to 0}\frac{\cos 2\theta - 1}{\theta\sin 5\theta}$

Working: (a) $\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = \frac{2\theta}{\theta} = 2$

E.g. 4 (a) Differentiate $f(x) = \sin x$ from first principles.
(b) Differentiate $f(x) = \cos x$ from first principles.

Hint: the formula for differentiation from first principles is $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{x+h-x}$

Working: (a)

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{x+h-x}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \quad \textit{compound identity}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h} - \lim_{h\to 0}\frac{\sin x}{h} \quad \textit{separate terms}$$

$$= \lim_{h\to 0}\frac{\sin x\times\left(1-\frac{1}{2}h^2\right)}{h} + \lim_{h\to 0}\frac{\cos x\times h}{h} - \lim_{h\to 0}\frac{\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x}{h} - \lim_{h\to 0}\frac{\frac{1}{2}h^2\sin x}{h} + \lim_{h\to 0}\frac{\cos x\times h}{h} - \lim_{h\to 0}\frac{\sin x}{h}$$

$$= -\sin x\times\lim_{h\to 0}\frac{\frac{1}{2}h^2}{h} + \cos x\times\lim_{h\to 0}\frac{h}{h}$$

$$= -\sin x\times\lim_{h\to 0}\frac{1}{2}h + \cos x\times 1$$

$$= -\sin x\times 0 + \cos x\times 1$$

$$= \cos x$$

E.g. 5 Find an approximation for the expansion $\dfrac{1+\sin\theta}{5+3\tan\theta-4\cos\theta}$ when θ is small i.e. ignore all terms beyond θ^2.

Working:

$$\begin{aligned}\frac{1+\sin\theta}{5+3\tan\theta-4\cos\theta} &\approx \frac{1+\theta}{5+3\theta-4\left(1-\frac{1}{2}\theta^2\right)}\\ &= \frac{1+\theta}{1+3\theta+2\theta^2}\\ &= \frac{1+\theta}{(1+\theta)(1+2\theta)}\\ &= \frac{1}{1+2\theta}\\ &= (1+2\theta)^{-1}\\ &= 1+(-1)\times 2\theta+\frac{-1\times-2}{2!}(2\theta)^2+\dots\\ &\approx 1-2\theta+4\theta^2\end{aligned}$$

Video: Small angle approximations

Solutions to Starter and E.g.s

Exercise
p158 7F Qu 1i, 2i, 3i, 4-6, 8-11, (12-13 red)

Summary
The small angle approximations are:

$$\sin\theta\approx\theta \qquad \tan\theta\approx\theta \qquad \cos\theta\approx 1-\frac{1}{2}\theta^2$$

The small angle approximations are reasonable up to about 0.35^c i.e. 20°