The Newton-Raphson Method

Starter

1. **(Review of last lesson)**
   For the equation $x^2 + 2^x - 9 = 0$ show that $x = 2.14$ is a solution to 2 d.p..

Notes

Iterative methods for solving equations or finding roots

Last lesson we used the sign change method to find two consecutive integers between which the root of an equation lies. We now need to find the value of the root i.e. to solve the equation.

We use iterative methods when we cannot solve an equation algebraically — therefore, we wouldn't solve a quadratic equation using an iterative method since we have the quadratic formula. The trial and improvement method we used at GCSE can be slow and laborious, especially when a high degree of accuracy is required (e.g. 3 d.p. or higher). So mathematicians use iterative methods to get to the answer quicker.

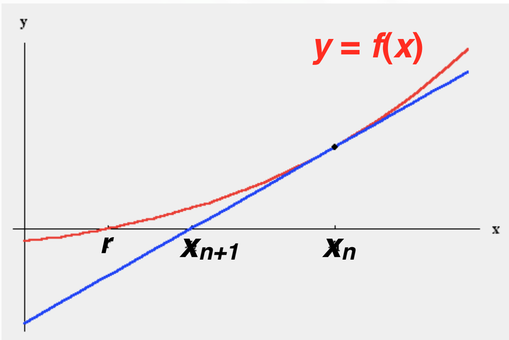
An iterative method repeatedly substitutes successive numbers into the same formula, hoping to get closer and closer to the root. For example, we substitute the value x_0 into a formula and the output is x_1. We then substitute x_1 in and get x_2 out. We repeat this method until successive values are the degree of accuracy that we want

Newton-Raphson method

One example of an iterative method that is used to solve equations (i.e. find the root of an equation) is the Newton-Raphson method (named after Sir Isaac Newton and Joseph Raphson). The N-R method uses differentiation to find the tangent to a function at a point.

Deriving the formula for the Newton-Raphson method

In the picture above, the curve in red, $y = f(x)$, meets the $x-$axis at r. This is the value we are trying to find.



Let x_n be the current estimate for r.

The tangent line to $y = f(x)$ at the point $\left(x_n, f(x_n)\right)$ has equation: $\quad y - f(x_n) = f'(x_n)(x - x_n)$

N.B. This comes from $y - y_1 = m(x - x_1)$.

Let x_{n+1} be the $x-$intercept of the tangent. In the Newton-Raphson method this is the ***next estimate*** of the root.

At this point, $y = 0$ and $x = x_{n+1}$ and we can substitute them into the equation above.

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

Rearranging gives: $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ this is the formula for the Newton-Raphson method.

N.B. The Newton-Raphson formula is given in the exam.
You do not need to be able to derive the formula.

Success criteria — Newton-Raphson method

1. Use the sign change method to find a suitable starting value, x_0, unless it is already given.
2. If you are given an equation, arrange it so that it equals zero and then put it equal to $f(x)$.
3. Find the derivative of the function.
4. Substitute the function and its derivative into the formula, replacing x by x_n.
5. Substitute x_0 into the formula to get x_1 — always work to at least 1 d.p. more than is required for the answer (e.g. if require answer to 4 d.p., use 5 d.p. in your working).
6. Repeat step 5 ***until successive iterations give the same value*** to the required accuracy.

N.B. The starting value is usually denoted by x_0 so x_1 is the 1st iteration and x_2 is the 2nd iteration etc.

E.g. 1 Use the Newton-Raphson method with a starting point of $x_0 = -2$ to find a root of the equation $f(x) = x^3 - 3x + 5$ correct to 5 decimal places.

Working: ***Find the derivative of the function:*** $f'(x) = 3x^2 - 3$

Substitute the function and its derivative into the formula:

$$x_{n+1} = x_n - \frac{x_n{}^3 - 3x_n + 5}{3x_n{}^2 - 3}$$

remember to replace x ***by*** x_n ***in the formula***

$$x_0 = -2$$

Substitute $x_0 = -2$ into formula: $x_1 = -2 - \dfrac{(-2)^3 - 3 \times (-2) + 5}{3 \times (-2)^2 - 3} = -\dfrac{7}{3}$

N.B. Make sure you use brackets around negative numbers in your calculation

Use $x_1 = -\dfrac{7}{3}$: $x_2 = -\dfrac{7}{3} - \dfrac{\left(-\frac{7}{3}\right)^3 - 3 \times \left(-\frac{7}{3}\right) + 5}{3 \times \left(-\frac{7}{3}\right)^2 - 3} = -2.2080555...$

N.B. Use the back arrow on your calculator so that you don't have to type out the whole calculation again. Then use the ANS button to input the new value into the formula (see the video below for details).

Substitute $x_2 = -2.2080555...$: $x_3 = ... = -2.279020...$
Substitute $x_3 = -2.279020...$: $x_4 = ... = -2.279018...$
Substitute $x_4 = -2.279018...$: $x_4 = ... = -2.279018...$
Since $x_4 = x_5$, this is the value of the root we require.
So $x = -2.27902$ (5 d.p.)

Video: How to use your calculator with Newton-Raphson calculations

E.g. 2 Write down the the Newton-Raphson iterative formula for finding the roots of $f(x) = 5x^2 - 6$.

E.g. 3 By using a starting value of $x_0 = 2.5$, use the Newton-Raphson method to find a solution of the equation $x^4 - 2x^3 = 5$ to 5 s.f.

E.g. 4 Solve the equation $x^3 + 3x^2 + 5x + 7 = 0$ using the Newton-Raphson method and starting with $x_0 = 1$. Give your answer to 4 s.f.

Video: Newton-Raphson method
Video: How to use your calculator with Newton-Raphson calculations

Newton-Raphson method EQ

Solutions to Starter and E.g.s

Exercise
Beware of examples involving special functions.
p307 14B 1a (i & ii), 2 (not bi), 3, 4, 6, 7
Hints: Qn 2bii: the derivative of e^{kx} is ke^{kx}
Qu 6: multiply the equation by x before finding the Newton-Raphson formula

Summary

The formula for the Newton-Raphson method is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$