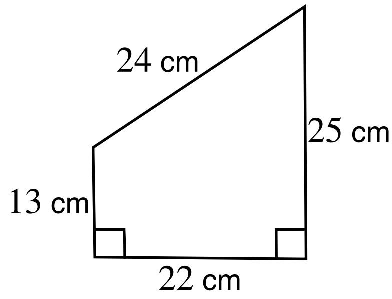
Trapezium Rule

Starter

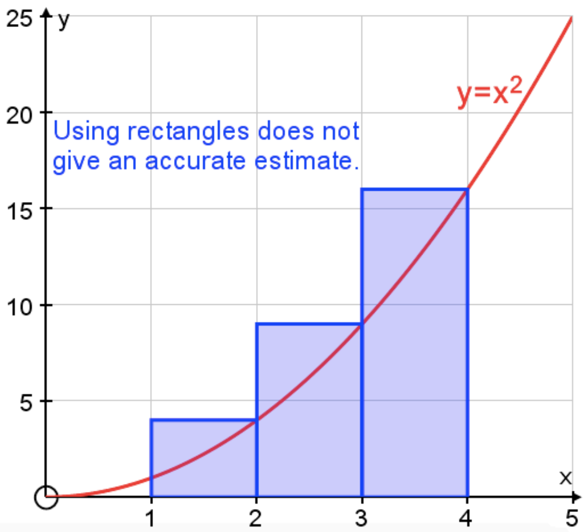
1. **(Review of last lesson)** Find the area of this shape.

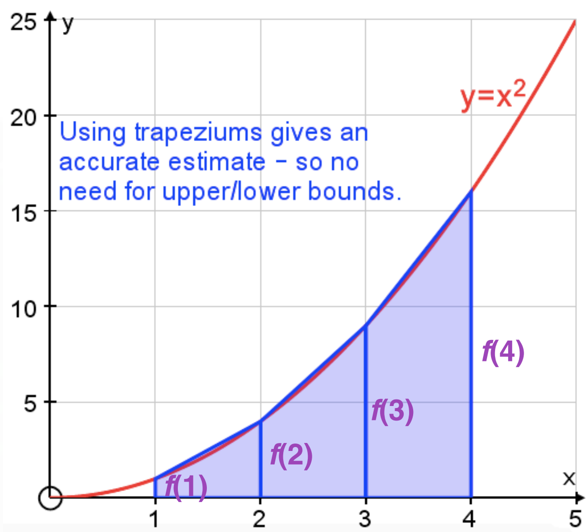


Notes

More accurate than using rectangles to estimate an integration (i.e. an area under a curve) is to use trapeziums (or trapezia).

Comparing the graphs for rectangles and trapeziums we can see that there is no need to do upper and lower bounds when using trapezia.





In the example above with the trapezia, the diagonal part of each trapezium is above the curve so the area found will be an overestimate.

Area of a trapezium $=$ half the sum of the parallel sides $\times$ the distance between them

In the example above we have three trapeziums whose area we have to find. The parallel sides are the vertical lines and the distance between the parallel sides is the "width" of the trapezium.

$$\text{First trapezium} = \frac{1}{2}\left(f(1)+f(2)\right)\times 1 = \frac{1}{2}\left(1^2+2^2\right)\times 1 = \frac{5}{2}$$

$$\text{Middle trapezium} = \frac{1}{2}\left(f(2)+f(3)\right)\times 1 = \frac{1}{2}\left(2^2+3^2\right)\times 1 = \frac{13}{2}$$

$$\text{Last trapezium} = \frac{1}{2}\left(f(3)+f(4)\right)\times 1 = \frac{1}{2}\left(3^2+4^2\right)\times 1 = \frac{25}{2}$$

$$\text{Total area of trapezia} = \frac{5}{2}+\frac{13}{2}+\frac{25}{2} = \frac{43}{2} = 21.5$$

So $\int_1^4 x^2 dx \approx 21.5$

Given that $\int_1^4 x^2 dx = 21$, our estimate is pretty close and we could improve it by using more trapezia.

Looking at the working, it can be seen that $f(2)$ and $f(3)$ appears twice in the calculation — e.g. $f(2)$ appears in the calculation for the first and the middle trapezia, while $f(3)$ appears in the calculation for the middle and the last trapezia. This is because they are the parallel sides of the trapezia that are on the inside. The outside values of $f(1)$ and $f(4)$ only appear once.

By adding the are of the trapezia together and taking $\frac{1}{2}$ and 1 out as factors we get

$$\text{Area} = \frac{1}{2} \times 1 \times \left[f(1) + 2f(2) + 2f(3) + f(4)\right]$$

width of trapezia

left-hand parallel side of first trapezium

parallel sides used with 2 trapezia

right-hand parallel side of last trapezium

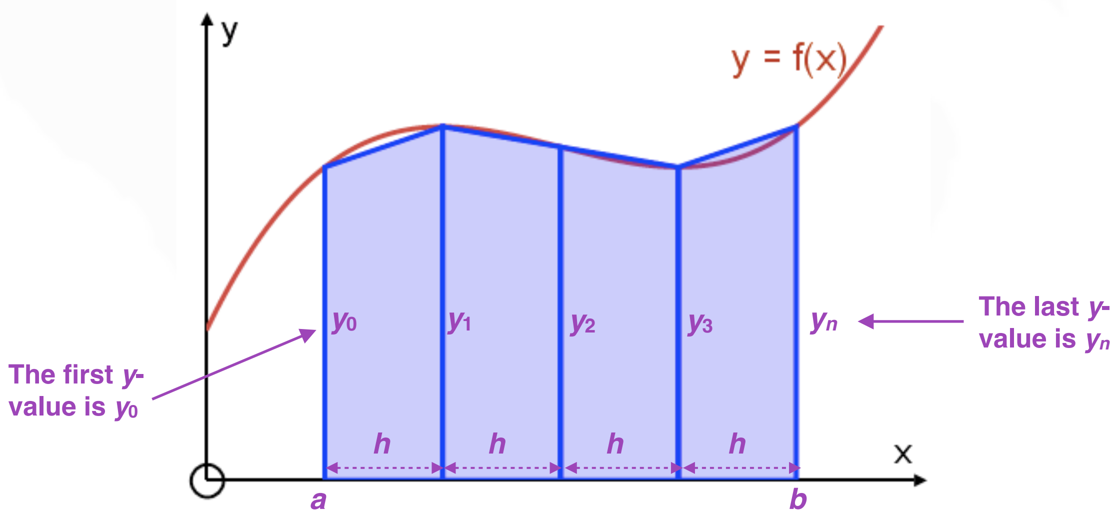
Remember: each $f(1), f(2)$ are the $y-$values of the function

Formula for the trapezium rule

The area given by $\int_a^b f(x)dx$ can be ***approximated*** using the trapezium rule.

Formula in words:

$$\text{Area} \approx \frac{1}{2} \times \text{width of trapezia} \times \left(\text{1st y-value} + 2(\text{middle y-values}) + \text{last y-value}\right)$$



Remember, a and b come from the integral $\int_a^b f(x)dx$

Formula in symbols

$$\text{Area} \approx \frac{h}{2}\Big(y_0 + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) + y_n\Big)$$

where $h = \dfrac{b-a}{n}$ is the trapezium width

N.B. To make our estimate more accurate we increase n i.e. increase the number of trapezia. A table of values is useful.

E.g. 1 Use the trapezium rule with 4 strips (i.e. trapezia) to approximate $\int_0^{0.8} \cos x\,dx$, giving your answer to 3 s.f.

N.B. Make sure your calculator is in ***radians***.

Working: Trapezia width $= \dfrac{0.8-0}{4} = 0.2$

$x-$values: start at $x = 0$ since this is the lower limit of the integral.
Add 0.2 each time to the $x-$value because this is the trapezia width
It can be useful to form a table to write down the values

x	0	0.2	0.4	0.6	0.8
y	cos 0	cos 0.2	cos 0.4	cos 0.6	cos 0.8

Area $\approx \dfrac{1}{2} \times 0.2 \times \left(\cos 0 + 2(\cos 0.2 + \cos 0.4 + \cos 0.6) + \cos 0.8\right)$
$= 0.715$ (3 s.f.)

E.g. 2 Use the trapezium rule with 3 strips (i.e. trapezia) to approximate $\int_1^{7} \ln x\,dx$, giving your answer exactly in the form $\ln a$ where a is an integer.

Percentage error

Percentage error is given by:

$$\text{Percentage error} = \frac{\left|\text{real value} - \text{estimate}\right|}{\text{real value}} \times 100\,\%$$

E.g. 3 Find an approximation for $\int_0^{0.6} x^2 dx$ using 3 strips with the trapezium rule and find the percentage error for the approximation given that the actual value is 0.072.

Working: Trapezia width $= \dfrac{0.6-0}{3} = 0.2$

$x-$values: start at $x = 0$ since this is the lower limit of the integral.
Add 0.2 each time to the $x-$value because this is the trapezia width

x	0	0.2	0.4	0.6
y	0^2	0.2^2	0.4^2	0.6^2

Area $\approx \dfrac{1}{2} \times 0.2 \times \left(0^2 + 2(0.2^2 + 0.4^2) + 0.6^2\right)$
$= 0.076$

$$\text{Percentage error} = \frac{\left|\text{real value} - \text{estimate}\right|}{\text{real value}} \times 100\,\%$$

$$\text{Percentage error} = \frac{\left|0.072 - 0.076\right|}{0.072} \times 100\,\%$$
$= 5.56\,\%$ (3 s.f.)

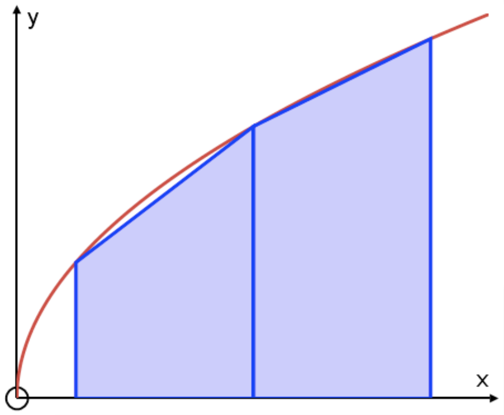
E.g. 4 Use the trapezium rule with 4 intervals to evaluate $\int_0^4 e^x dx$ to 4 s.f. Given that the correct value is $e^4 - 1$, find the percentage error of your value to 1 d.p..

Under and overestimates using the trapezium rule

When the curve is below the trapezia, the trapezium rule will give an overestimate. Conversely, if the curve is above the diagonal lines of the trapezia, the trapezium rule will give an underestimate.

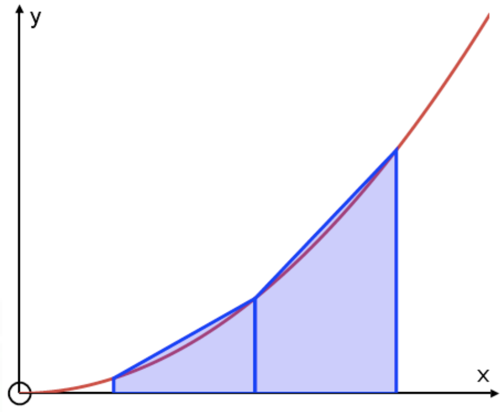
Underestimate

Curve is above the trapezia



Overestimate

Curve is below the trapezia



Video: Trapezium rule (including under-and over-estimates)

Trapezium rule EQ

Solutions to Starter and E.g.s

Exercise

p339 15B Qu 1aci, 3ace, 4-7

Summary

The area given by $\int_a^b f(x)dx$ can be ***approximated*** using the trapezium rule by

Area $\approx \frac{1}{2} \times$ width of trapezia $\times$ $\Big($1st y-value + 2(middle y-values) + last y-value$\Big)$

Area $\approx \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) + y_n\right)$

where $h = \frac{b-a}{n}$ is the trapezium width

To make our estimate more accurate we increase n i.e. increase the number of trapezia