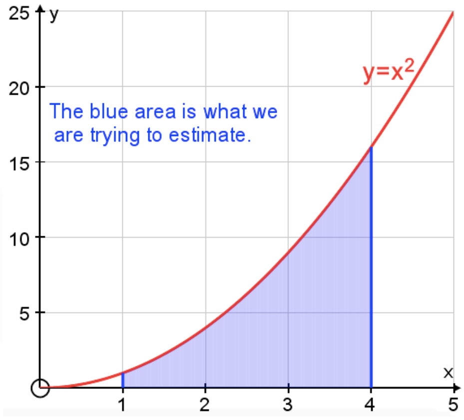
Upper and Lower Bounds of Integration

Notes

Consider the area under the curve $f(x) = x^2$ between $x = 1$ and $x = 4$.

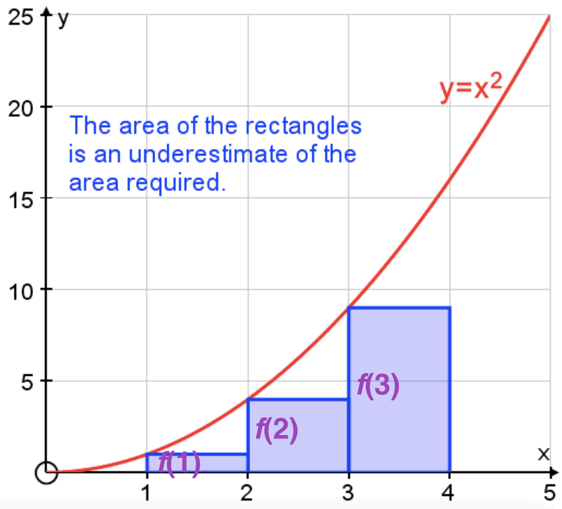
The blue part is the area we are trying to estimate. Of course, we can use integration for this function in order to find the exact value of the area but some functions cannot be integrated so we have to to estimate the area.



We can ***estimate*** the area by splitting the area into three rectangles. There are two ways to do this — one way gives an underestimate and one way gives an overestimate.

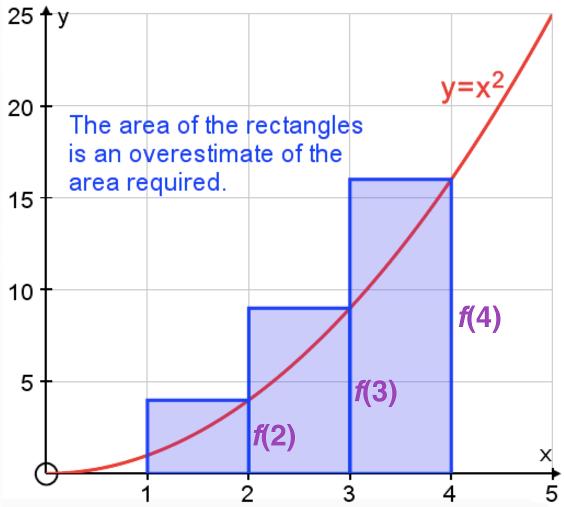
Underestimate (*lower bound*)

Each rectangle misses out some of the required area and so the total gives an underestimate.



Overestimate (*upper bound*)

Each rectangle has extra area so the total gives an overestimate,



We know that the actual area lies between the under- and overestimate so these could be considered ***lower and upper bounds*** for the actual area.

The height of each rectangle is $f(x)$ where x is the left-hand or right-hand value depending on whether the rectangle is part of an under- or overestimate.

In this example, $f(x) = x^2$.

$$\begin{aligned}\text{Lower bound} &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3)\\ &= 1 \times 1^2 + 1 \times 2^2 + 1 \times 3^2\\ &= 14\end{aligned}$$

1 is the width of each rectangle

$$\begin{aligned}\text{Upper bound} &= 1 \times f(2) + 1 \times f(3) + 1 \times f(4)\\ &= 1 \times 2^2 + 1 \times 3^2 + 1 \times 4^2\\ &= 29\end{aligned}$$

So from: lower bound < integral < upper bound

we get: $$14 < \int_1^4 x^2 dx < 29$$

In fact, $\int_1^4 x^2 dx = 21$

N.B. We can obviously make our lower and upper bounds more accurate by using more rectangles — in this case the rectangles would become narrower.

The not-so-useful formulae

For upper and lower bounds of the integral $\int_a^b f(x)dx$, n rectangles gives width $h = \frac{b-a}{n}$

I include the formulae but I recommend you understand the method as it is difficult to remember them.

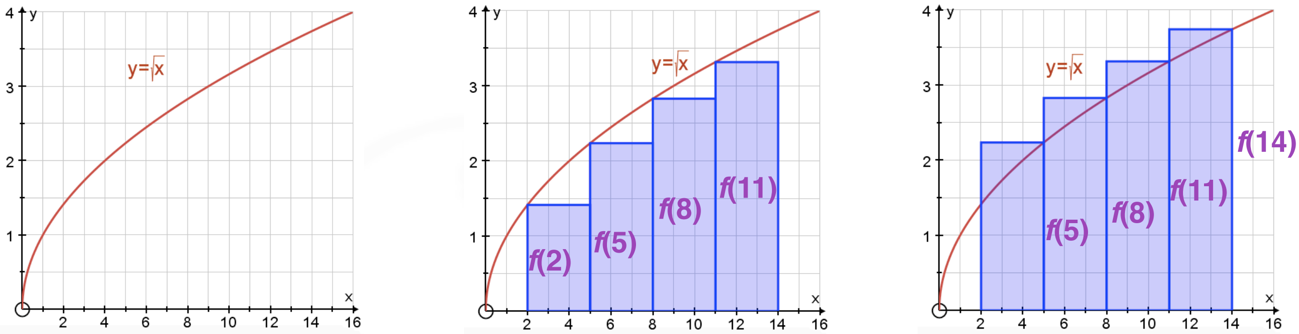
$$\text{Lower bound} = \frac{b-a}{n}\Big(f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\Big)$$

$$\text{Upper bound} = \frac{b-a}{n}\Big(f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+(n-1)h) + f(b)\Big)$$

N.B. Each rectangle has the same width, i.e. $\frac{b-a}{n}$.
More useful than the formulae is a diagram.

E.g. 1 Find upper and lower bounds for the integration $\int_2^{14} \sqrt{x}\,dx$, using 4 rectangles, giving each answer to 3 s.f.

Working:



4 rectangles to fit between $x = 2$ and $x = 14$

So the width of each rectangle is $\dfrac{14-2}{4} = 3$

Lower bound $= 3 \times f(2) + 3 \times f(5) + 3 \times f(8) + 3 \times f(11)$
$= 3 \times \sqrt{2} + 3 \times \sqrt{5} + 3 \times \sqrt{8} + 3 \times \sqrt{11}$
$= 29.4$ (3 s.f.)

Upper bound $= 3 \times f(5) + 3 \times f(8) + 3 \times f(11) + 3 \times f(14)$
$= 3 \times \sqrt{5} + 3 \times \sqrt{8} + 3 \times \sqrt{11} + 3 \times \sqrt{14}$
$= 36.4$ (3 s.f.)

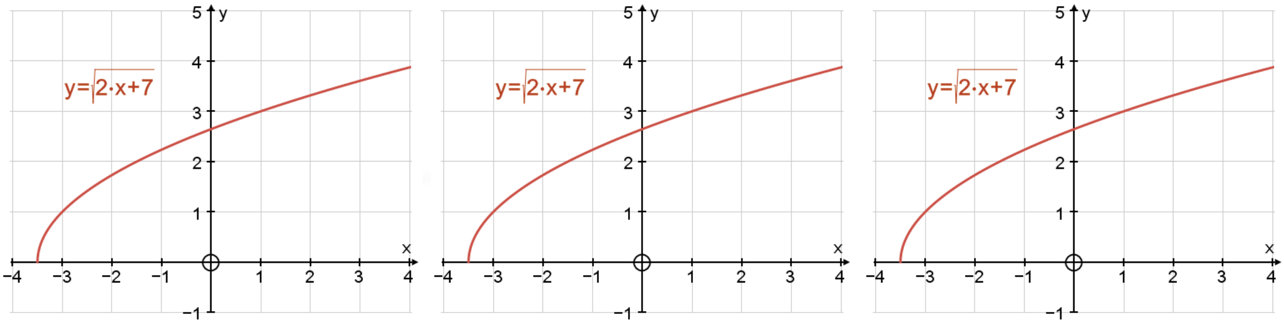
So $\quad 29.4 < \int_2^{14} \sqrt{x}\,dx < 36.4$

E.g. 2 Use two rectangles to give upper and lower bounds for $\int_1^5 e^x\,dx$, giving each answer to 3 s.f..

Working:

E.g. 3 Use three rectangles to give upper and lower bounds for $\int_{2}^{3.5} \sqrt{2x+7}\,dx$

Working:



Video: Riemann approximation

Video: Numerical integration with rectangles

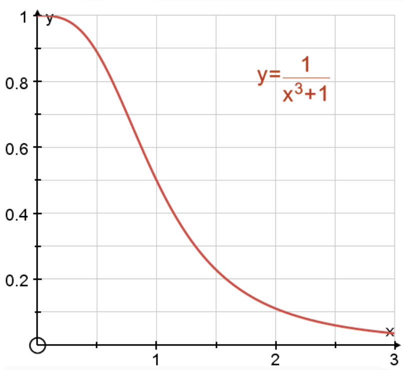
Riemann approximation (notes)

Solutions to Starter and E.g.s

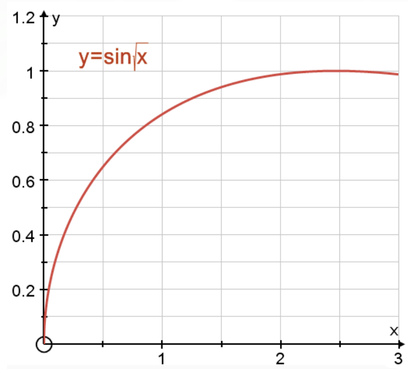
Exercise

p334 15A Qu 1aii, 1bi (use radians), 1cii (use radians), 3, 4, 6b

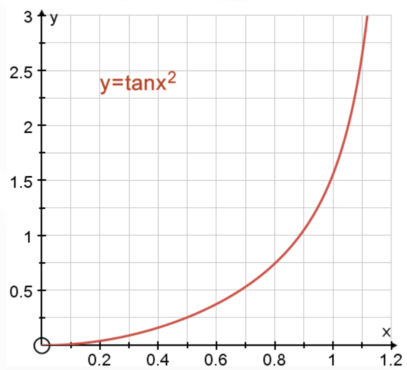
Graph for 1aii



Graph for 1bi



Graph for 1cii



Summary

Remember: diagrams are for more useful than the formulae.

For upper and lower bounds of the integral $\int_{a}^{b} f(x)dx$, n rectangles gives width $h=\frac{b-a}{n}$

Lower bound $=\frac{b-a}{n}\left(f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h)\right)$

Upper bound $=\frac{b-a}{n}\left(f(a+h)+f(a+2h)+f(a+3h)+\ldots+f(a+(n-1)h)+f(b)\right)$