Using Z—scores and the standard distribution

Starter

1. For $X \sim \text{N}(165, 8^2)$, use your calculator to find:
   (a) $P(X < 160)$
   (b) $P(X \geq 157)$
   (c) $P(X \leq 162, X > 170)$
   (d) $P(|X - 165| < 14)$
   Give your answers to 4 d.p..

Notes

To calculate $P(151 < X < 179)$ the area is given by the integral $\displaystyle\int_{151}^{179} \frac{1}{8\sqrt{2\pi}} e^{-\frac{(x-165)^2}{32}} dx$.

To calculate its value requires complicated, non-standard integration techniques and would require a separate calculation for every different value of the mean and standard deviation.

Therefore, statisticians developed the ***standard normal distribution***, Z, where the mean is 0 and the standard deviation is 1 i.e. $Z \sim \text{N}(0, 1)$. The complicated calculations for each area under the curve was performed just once and recorded in sets of statistical tables.

Normal distributions of the form $X \sim \text{N}(\mu, \sigma^2)$ were transformed to $Z \sim \text{N}(0, 1)$ using the formula $z = \dfrac{x - \mu}{\sigma}$.

E.g. $X \sim \text{N}(165, 8^2)$ can be transformed by $z = \dfrac{x - 165}{8}$ so that:

$$P(151 < X < 179) = P\left(\frac{151 - 165}{8} < Z < \frac{179 - 165}{8}\right) = P(-1.75 < Z < 1.75)$$

With the advent of calculators that can find the areas under the normal curve, the formula $z = \dfrac{x - \mu}{\sigma}$ sees less use than it used to.

E.g. 1 Find the corresponding $Z-$value to the given value of X:
(a) $X \sim \text{N}(18, 4)$, $x = 21$
(b) $X \sim \text{N}(256, 9^2)$, $x = 238$

Working: (a) $z = \dfrac{x - \mu}{\sigma}$: $z = \dfrac{21 - 18}{2} = 1.5$

E.g. 2 The time taken for students aged 11 to run 100 m can be considered as having a normal distribution with a mean of 15.6 seconds and a standard deviation of 0.4 seconds. Find the probability that the time taken for a student to complete the race is:
(a) under 15 seconds
(b) at least 16 seconds
(c) between 15 and 16 seconds.
Give your answers to 4 s.f.

Working: (a) $X \sim \text{N}(15.6, 0.4^2)$
$P(X < 15) = 0.006210$

Questions involving conditional probability

The probability of event B happening, given that event A has already happened is denoted by $P(B|A)$.

The formula for $P(B|A)$ is: $$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

E.g. 3 Scores on a test are normally distributed with a mean of 68 and a standard deviation of 8.

(a) Find the probability that a student scored:

(i) at least 75 on the test

(ii) at least 75 on the test given that the student scored at least 70 on the test.

(b) In a group of 50 students, how many students would you expect to score between 65 and 72 on the test. Give your answer to the nearest number of students.

Working: (a) (i) $X \sim \text{N}(68, 8^2)$

$P(X \geq 75) = 0.1908$

The probability a student scored at least 75 is 0.1908 (4 s.f.)

(ii)
$$\begin{aligned} P(X \geq 75 | X \geq 70) &= \frac{P(X \geq 75 \cap X \geq 70)}{P(X \geq 70)} \\ &= \frac{P(X \geq 75)}{P(X \geq 70)} \\ &\approx \frac{0.19079}{0.40129} \\ &= 0.4754 \end{aligned}$$

The probability a student scored at least 75 given that they scored at least 70 is 0.4754 (4 s.f.)

(b) Number $= 50 \times P(65 < X < 72) = 50 \times 0.3376 = 16.9$

The number of students would you expect to score between 65 and 72 on the test is 17.

E.g. 4 If X is a normally distributed variable with a mean of 24 and a standard deviation of 2, find:

(a) $P(X > 28 | X > 26)$ (b) $P(26 < X < 28 | X > 27)$

Give your answers to 3 s.f..

Video: Standard normal distribution

Video: Probability using tables

Solutions to Starter and E.g.s

Exercise

p384 17B Qu 1i, 2i, 3-10, (11-14 red)

Summary

The standard normal distribution, Z, has mean of 0 and standard deviation of 1 i.e. $Z \sim \text{N}(0, 1)$.

Normal distributions of the form $X \sim \text{N}(\mu, \sigma^2)$ are transformed to $Z \sim \text{N}(0, 1)$ using the formula $z = \dfrac{x - \mu}{\sigma}$.