Write yours and your teacher's name at the top of your answer sheets.

U6 Mathematics Mock

Paper 1 (X – JW/TDM/VMP/NDW)
February 2023
2022-2023

Duration: 1 hour 30 minutes

Total number of marks: 69

Write your answers on file paper.

You are permitted to use a scientific or graphical calculator in this paper.

Final answers should be given to a degree of accuracy appropriate to the context.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation from first principles

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Integration

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1} + c$

Integration by parts $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k+\frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $\mathrm{P}(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u+v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

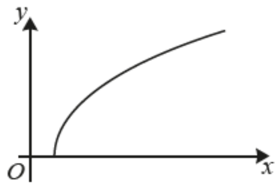
$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

1.



The diagram shows part of the curve $y=\sqrt{x^2-1}$.

(a) Use the trapezium rule with 4 intervals to find an estimate for $\int_1^3 \sqrt{x^2-1}\,dx$.

Give your answer correct to **3** significant figures. **[4]**

(b) State whether the value from part **(a)** is an under-estimate or an over-estimate, giving a reason for your answer. **[1]**

(c) Explain how the trapezium rule could be used to obtain a more accurate estimate. **[1]**

2.

In this question you must show detailed reasoning.

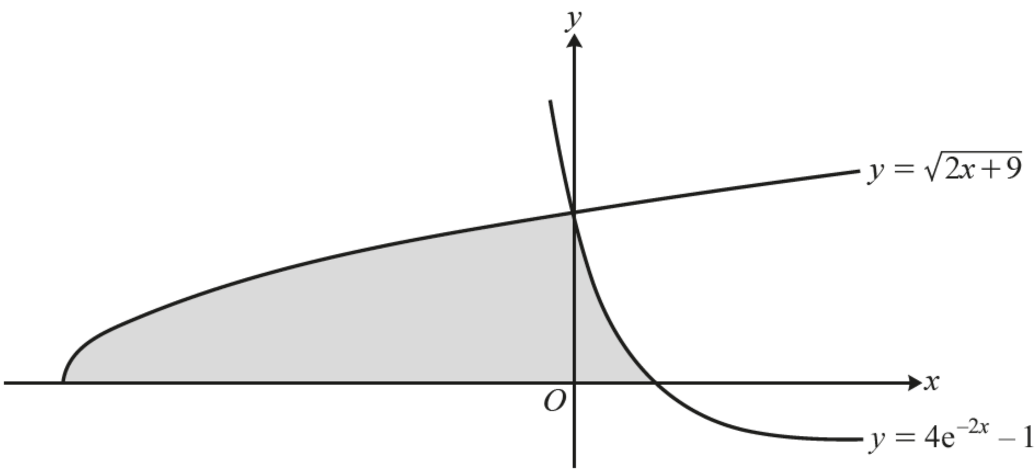
A curve has equation $y=x^3-3x^2+4x$.

(a) Show that the curve has no stationary points. **[2]**

(b) Show that the curve has exactly one point of inflection. **[2]**

3.

In this question you must show detailed reasoning.

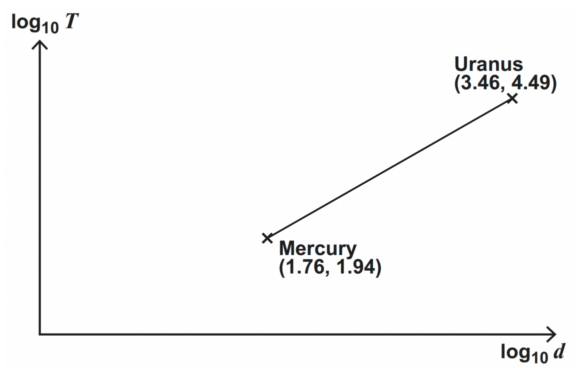


The diagram shows the curves $y=\sqrt{2x+9}$ and $y=4e^{-2x}-1$ which intersect on the y-axis. The shaded region is bounded by the curves and the x-axis.

Determine the area of the shaded region, giving your answer in the form $p+q\ln 2$ where p and q are constants to be determined. **[8]**

4.
A planet takes T days to complete one orbit of the Sun. T is known to be related to the planet's average distance d, in millions of kilometres, from the Sun.

A graph of $\log_{10} T$ against $\log_{10} d$ is shown with data for Mercury and Uranus labelled.



(a) (i) Find the equation of the straight line in the form $\log_{10} T = a + b \log_{10} d$, where a and b are constant to be found.

(3)

(ii) Show that $T = Kd^n$ where K and n are constant to be found

(2)

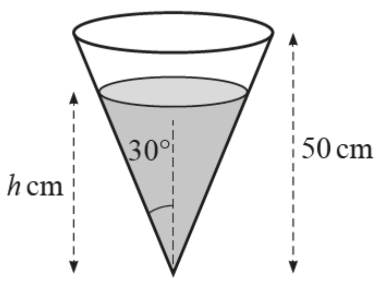
(b) Neptune takes approximately 60 000 days to complete one orbit of the Sun.
Use your answer to (a)(ii) to find an estimate for the average distance of Neptune from the Sun.

(2)

5.
A curve has equation $2x^3 + 6xy - 3y^2 = 2$.

Show that there are no points on this curve where the tangent is parallel to $y = x$. **[8]**

6.



The diagram shows a water tank which is shaped as an inverted cone with semi-vertical angle 30° and height 50 cm. Initially the tank is full, and the depth of the water is 50 cm.

Water flows out of a small hole at the bottom of the tank. The rate at which the water flows out is modelled by $\frac{dV}{dt} = -2h$, where $V\,\text{cm}^3$ is the volume of water remaining and h cm is the depth of water in the tank t seconds after the water begins to flow out.

Determine the time taken for the tank to become empty.

[For a cone with base radius r and height h the volume V is given by $\frac{1}{3}\pi r^2 h$.] **[7]**

7.

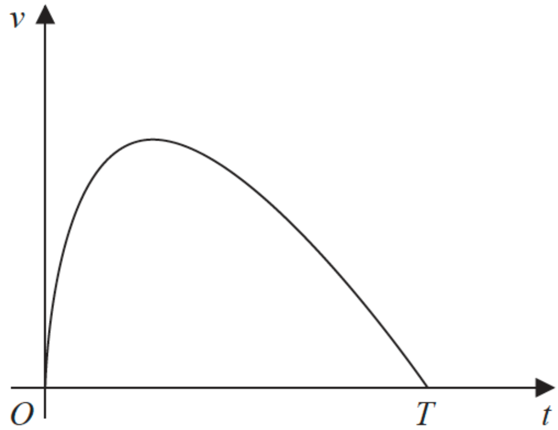


Figure 2

A car stops at two sets of traffic lights.

Figure 2 shows a graph of the speed of the car, $v\,\text{ms}^{-1}$, as it travels between the two sets of traffic lights.

The car takes T seconds to travel between the two sets of traffic lights.

The speed of the car is modelled by the equation

$$v = (10 - 0.4t)\ln(t + 1) \qquad 0 \leqslant t \leqslant T$$

where t seconds is the time after the car leaves the first set of traffic lights.

According to the model,

(a) find the value of T

(1)

(b) show that the maximum speed of the car occurs when

$$t = \frac{26}{1 + \ln(t + 1)} - 1$$

(4)

Using the iteration formula

$$t_{n+1} = \frac{26}{1 + \ln(t_n + 1)} - 1$$

with $t_1 = 7$

(c) (i) find the value of t_3 to 3 decimal places,

(ii) find, by repeated iteration, the time taken for the car to reach maximum speed.

(3)

8.

Use the substitution $x = 2\sin\theta$ to show that $\int_1^{\sqrt{3}} \sqrt{4-x^2}\,dx = \frac{1}{3}\pi$. **[7]**

9.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

10.

The gradient function of a curve is given by $\frac{dy}{dx} = \frac{3x^2 \ln x}{e^{3y}}$.

The curve passes through the point (e, 1).

(a) Find the equation of this curve, giving your answer in the form $e^{3y} = f(x)$. **[6]**

(b) Show that, when $x = e^2$, the y-coordinate of this curve can be written as $y = a + \frac{1}{3}\ln(be^3 + c)$, where a, b and c are constants to be determined. **[3]**