U6 Ma Mock Teacher X 22-23 SOLUTIONS [69]

1.

(a)		$0.5\times0.5\left\{0+2\sqrt{2}+2\left(\frac{\sqrt{5}}{2}+\sqrt{3}+\frac{\sqrt{21}}{2}\right)\right\}$	B1	1.1a	State the 4 correct non-zero y-values and no others	Exact values (including unsimplified) or decimal equivs (0, 1.12, 1.73, 2.29, 2.83) – 3sf or better B0 if other ordinates seen unless clearly not intended to be used
			M1*	1.1a	Attempt to find area between $x=1$ and $x=3$, using $k\{y_0+y_n+2(y_1+\ldots+y_{n-1})\}$	Big brackets need to be seen or implied y-values must be correctly placed Must be using attempts for at least 4 y-values (but no need to see $y=0$ explicitly) Condone using other than 4 intervals as long as values equally spaced between $x=1$ and $x=3$
			M1d*	1.1a	Use $k=0.5\times0.5$ soi	Dep on previous M1 Or using $k=0.5h$, h consistent with their different number of intervals
		$=3.28$	A1 [4]	1.1	Obtain 3.28, or better	Allow answers to > 3sf, as long as they round to 3.28
(b)		Under-estimate, as the tops of the trapezia are below the curve	B1 [1]	3.2b	Under-estimate, with any valid explanation	Condone just 'trapezia under curve' Or curve is concave / decreasing gradient (not decreasing function) Accept explanation on diagrams Allow comparing to true value (3.36) B0 if any additional incorrect or contradictory statements
(c)		Use more trapezia, of a lesser width, between the same limits	B1 [1]	3.2b	Convincing reason	Condone just 'more trapezia' or 'narrower trapezia' Could refer to strips or intervals

2.

(a)		**DR** $\frac{dy}{dx}=3x^2-6x+4=0$ $b^2-4ac=-12$ or D=−12 or $3(x-1)^2+1=0$ oe No (real) roots or no value of x, or can't $\sqrt{\text{negative}}$ or gradient always +ve.	M1 A1 [2]	3.1a 1.1	Differentiate & equate to 0. May be implied by calc of D or $x=\frac{6\pm\sqrt{36-48}}{6}$ or $x=\frac{6\pm i\sqrt{12}}{6}$ oe Must see justification as line above, no errors, <u>&</u> statement Other correct forms of the quadratic equation and justification may be seen.
(b)		**DR** $\frac{d^2y}{dx^2}=6x-6=0$ $x=1$ gives a point of inflection or $x=1$ & show that, either side of this point, gradient does not change sign <u>or</u> second derivative does change sign	M1 A1 [2]	1.1 2.2a	Differentiate their $\frac{dy}{dx}$ and = 0. Can be implied by $x=1$ Statement "$x=1$ gives a point of inflection" is enough. or This equation has one root. (so curve has one inflection) Not just "$x=1$" Ignore y-coordinate

3.

DR	M1	2.1	**M1** for $k(2x+9)^{\frac{3}{2}}$ with non-zero k	$k \neq 1$
$\int (2x+9)^{\frac{1}{2}}\,dx = \frac{1}{3}(2x+9)^{\frac{3}{2}}$	A1	1.1	cao (allow unsimplified)	
$\left[\frac{(2x+9)^{\frac{3}{2}}}{3}\right]_{-\frac{9}{2}}^{0} = 9$	A1	1.1	Uses correct limits (or implies correct limits) to get 9. Condone limits the wrong way round leading to -9 but must be changed to $+9$	
$4e^{-2x} - 1 = 0 \Rightarrow e^{-2x} = \frac{1}{4}$ $-2x = \ln\left(\frac{1}{4}\right)$	M1*	3.1a	Attempt to solve $4e^{-2x} - 1 = 0$ by correctly taking logs of both sides leading to $\pm 2x = \pm \ln \alpha$ where $\alpha > 0$	Allow sign errors and other minor slips only
$x = -\frac{1}{2}\ln\left(\frac{1}{4}\right)$	A1	1.1	Or equivalent exact value (soi possibly by correct exact value used later)	e.g. $\frac{1}{2}\ln 4$ or $\ln 2$
$\int (4e^{-2x} - 1)\,dx = -2e^{-2x} - x$	M1*	1.1	Integrate $4e^{-2x} - 1$ to obtain $ce^{-2x} \pm x$	Where c is non-zero and $c \neq 4$
$\int_0^{\frac{1}{2}\ln 4} (4e^{-2x} - 1)\,dx = \left(-2e^{-\ln 4} - \frac{1}{2}\ln 4\right) - (-2)$	M1dep*	1.1	Uses limits correctly $F(\frac{1}{2}\ln 4)$- $F(0)$ (with their $\frac{1}{2}\ln 4$) – **dependent on the previous two M marks** (allow non-exact top limit). Condone limits the wrong way round only if the sign of their answer is subsequently changed	If zero limit is assumed to give 0 (with no working) then **M0**
Area = $9 + \frac{3}{2} - \frac{1}{2}\ln 4 = \frac{21}{2} - \frac{1}{2}\ln 4 = \frac{21}{2} - \ln 2$	A1	2.2a	**If the values of the integral(s) are changed from negative to positive (e.g. from limits the wrong way round) with no justification given then A0**	p and q need not be explicitly stated. $p = \frac{21}{2}$ (oe) and $q = -1$
	[8]			

4.

(a)(i)	Forms a correct expression for the gradient or sets up two correct simultaneous equations PI by a = –0.7 or b = 1.5 Ignore missing labels	1.1a	M1	$\frac{4.49-1.94}{3.46-1.76}=1.5$ $\log_{10} T - 1.94 = 1.5(\log_{10} d - 1.76)$ $\log_{10} T = -0.7 + 1.5\log_{10} d$
	Obtains a = –0.7 or b = 1.5 OE Ignore missing labels	1.1b	A1	
	Obtains a = –0.7 and b = 1.5 or seen in the logarithmic equation ISW	1.1b	A1	
	Subtotal		**3**	
(a)(ii)	Uses one law of logarithm correctly Allow use of original equation without values for a and b If values are used, $a \neq 0$	3.3	M1	$\log_{10} T - \log_{10} d^{1.5} = -0.7$ $\log_{10}\left(\frac{T}{d^{1.5}}\right) = -0.7$ $\frac{T}{d^{1.5}} = 10^{-0.7}$ $T = 10^{-0.7} \times d^{1.5}$
	Completes reasoned argument to obtain $T = \text{K}d^n$ with K = $10^{-0.7}$ or AWRT 0.2 and n = 1.5 ISW Must come from correct working	2.1	R1	
	Subtotal		**2**	
(b)	Forms an equation using their answer to **(a)(ii)** with K > 0 and n > 0 and $T = 60000$ Must only have unknown d in the equation	3.4	M1	$60000 = 0.2 \times d^{1.5}$ $d = 4488.5$ Average distance is approximately 4500 million kilometres
	Obtains AWRT 4500 million kilometres ACF with units For example • 4.5×10^9 **kilometres** • 4500×10^6 **kilometres** • 4.5×10^{12} **metres** • 4500×10^9 **metres**	3.2a	A1	
	Subtotal		**2**	

5.

$6x^2 + 6y + 6x\frac{dy}{dx} - 6y\frac{dy}{dx} = 0$	**M1**	**1.1a**	Attempt implicit differentiation	Either of the two $\frac{dy}{dx}$ terms correct, allowing sign errors Condone $6x^2dx + 6ydx + 6xdy - 6ydy$
	B1	**1.1a**	Use product rule correctly on middle term	Both terms correct Must now be $6y + 6x\frac{dy}{dx}$, or implied in a correct expression for $\frac{dy}{dx}$
	A1	**1.1**	Obtain correct derivative on LHS	Condone missing or incorrect RHS Must now have $\frac{dy}{dx}$ and not just dy or dx in terms
$6x^2 + 6y + 6x - 6y = 0$	**M1**	**3.1a**	Use $\frac{dy}{dx} = 1$ in their equation	Must now be equation, but RHS could be incorrect (eg '= 2')
$x^2 + x = 0$ $x = 0, x = -1$	**B1**	**1.1a**	Solve correct quadratic in x to obtain two correct roots (possibly **BC**) Quadratic must come from correct implicit differentiation	**B0** if x 'cancelled' in quadratic to give $x = -1$ as only root, but **M1A1** still available
$x = 0$ gives $3y^2 = -2$, but y^2 has to be ≥ 0, so no solutions	**B1**	**2.3**	Explicitly reject $x = 0$, with reasoning $x = 0$ must come from $x^2 + x = 0$	eg negative numbers cannot be square rooted or $y^2 \neq -\frac{2}{3}$ 2 as y is real (just $y^2 \neq -\frac{2}{3}$ is insufficient) Must be sensible reason and not just 'math error' or 'not possible' Could say that there are only imaginary (or not real) roots – condone 'complex' roots
$x = -1$ gives $3y^2 + 6y + 4 = 0$ $b^2 - 4ac = 36 - 48 = -12$	**M1**	**2.1**	Attempt to determine the number of real roots of their 3 term quadratic in y	From substituting their x value into the equation of the curve Consider discriminant, or use quadratic formula, or attempt minimum value of function
$-12 < 0$ hence no (real) roots	**A1**	**2.4**	Obtain correct discriminant from correct quadratic and conclude appropriately $x = -1$ must come from $x^2 + x = 0$	If using quadratic formula then it must be fully correct and attention drawn to why there are no real roots
	[8]			

6.

Summary method: Express V in terms of h	**B1**	3.3	Correct substitution
Differentiate V with respect to h	**M1**	3.4	NOT if h = 50 or r = 50tan30 used
Attempt chain rule, Attempt separate variables	**M1** **M1**		Resulting equation must involve exactly 2 variables Their equation must involve exactly 2 variables
Correct integrals Substitute correct limits Answer	**A1** **M1** **A1**		Ignore limits Integrals must be of correct forms (see examples below) Note 1 Candidates who substitute numerical values for h or V or r may be able to score the 2nd and/or 3rd M1 marks, but probably nothing else. See the example of this below. Note 2. There is a special case for candidates who use $r = h\sin 30$ (answer $\frac{625\pi}{4}$ or 491). These can score all 4 M-marks and the final A1 Note 3. The chain rule may be used to find $\frac{dV}{dt}$ or $\frac{dh}{dt}$ or $\frac{dV}{dh}$ or $\frac{dV}{dr}$ or other derivatives. Two of the example methods below illustrate use of $\frac{dV}{dt}$ and $\frac{dV}{dr}$, but use of other derivatives can also lead to correct methods.
Example method 1 $V = \frac{\pi}{3}(h\tan 30^o)^2 h$ or $V = \frac{\pi}{3}\left(\frac{h}{\sqrt{3}}\right)^2 h$ oe	**B1**	3.3	or $V = \frac{\pi}{9}h^3$ oe
$\frac{dV}{dh} = \frac{\pi}{3}h^2$	**M1**	3.4	Attempt differentiate their V in terms of h only NOT if h = 50 or r = 50tan30 used.
$\frac{dV}{dt} = \frac{\pi}{3}h^2\frac{dh}{dt}$ oe or $\frac{dh}{dt} = \frac{3}{\pi h^2}\frac{dV}{dt}$	**M1**	2.1	Attempt use chain rule for $\frac{dV}{dt}$ or $\frac{dh}{dt}$ in terms of t & h only
$("\frac{\pi}{3}h^2\frac{dh}{dt}" = -2h$ oe or $\frac{dh}{dt} = \frac{-6}{\pi h})$			(Set their $\frac{dV}{dt} = -2h$)
$\pi\int_{50}^{0} h\,dh = -\int_{0}^{t} 6\,dt$ oe	**M1**	1.1	Attempt separate variables in their equation in terms of h and t only (not V or r). Integral signs not essential
$\left[\frac{\pi h^2}{2}\right]_{50}^{0} = \left[-6t\right]_0^t$ oe	**A1**	2.1	Correct integrals, any limits or none
$-\pi\times\frac{50^2}{2} = -6t$	**M1**	1.1	Substitute correct limits into integrals of forms ah^2 & bt OR substitute $t = 0$ & $h = 50$ to find c and substitute $h = 0$
Time = $\frac{625\pi}{3}$ secs or 654 secs (3 sf) oe	**A1**	3.4	Allow without secs or 10.9 mins or 10 mins 54 secs or: SC. Use of $r = h\sin 30$ (answer $\frac{625\pi}{4}$ or 491) can score all 4 M-marks and final A1
	[7]		

Example method 2			
$V = \frac{\pi}{3}r^2\frac{r}{\tan 30^{\circ}}$ or $V = \frac{\pi}{\sqrt{3}}r^3$ oe	**B1**		Subst $h = \frac{r}{\tan 30^{\circ}}$ into correct formula for V
$\frac{dV}{dr} = \sqrt{3}\pi r^2$	**M1**		
$\frac{dV}{dt} = \sqrt{3}\pi r^2 \frac{dr}{dt}$ oe $("\sqrt{3}\pi r^2 \frac{dr}{dt}" = -2r\sqrt{3}$ oe)	**M1**		Attempt use chain rule to find $\frac{dV}{dt}$ or $\frac{dr}{dt}$ in terms of t and r (Set their $\frac{dV}{dt} = -2r\sqrt{3}$ oe)
$\pi \int_{\frac{50}{\sqrt{3}}}^{0} r\,dr = -\int_{0}^{t} 2dt$ oe	**M1**		Attempt separate variables in their equation in terms of r and t only (not V or h). Integral signs not essential
$\left[\frac{\pi r^2}{2}\right]_{\frac{50}{\sqrt{3}}}^{0} = \left[-2t\right]_0^t$ oe	**A1**		Correct integrals, any limits or none
$-\frac{\pi \times 50^2}{6} = -2t$	**M1**		Substitute correct limits into integrals of the form ar^2 & bt OR substitute $t = 0$ & $r = \frac{50}{\sqrt{3}}$ to find c and substitute $r = 0$
Time $= \frac{625\pi}{3}$ secs or 654 secs (3 sf) oe	**A1**	**3.4**	Allow without secs or 10.9 mins or 10 mins 54 secs SC. Use of $r = h\sin 30$ (answer 491) can score M4A1

Example method 3 (NOT using chain rule)			**This method is different from the summary method above**
$V = \frac{\pi}{3}(h\tan 30^{\circ})^2 h$ or $V = \frac{\pi}{3}\left(\frac{h}{\sqrt{3}}\right)^2 h$ oe	**B1**	**3.3**	or $V = \frac{\pi}{9}h^3$ oe
$h = \sqrt[3]{\frac{9V}{\pi}}$	**M1**		Allow $h = kV^{1/3}$
$\frac{dV}{dt} = -2 \times \sqrt[3]{\frac{9V}{\pi}}$	**M1**		$\frac{dV}{dt} = -2\times$(their h in terms of V)
$\sqrt[3]{\frac{\pi}{9}} \int_{\frac{\pi 50^3}{9}}^{0} V^{-1/3}dV = -2\left[t\right]_0^t$	**M1**		Attempt separate variables in their equation in terms of V and t only (not h or r). Integral signs not essential
$\sqrt[3]{\frac{\pi}{9}} \times \frac{3}{2}\left[V^{2/3}\right]_{\frac{\pi 50^3}{9}}^{0} = -2t$	**A1**		Correct integrals, any limits or none
$-\sqrt[3]{\frac{\pi}{9}} \times \frac{3}{2} \times \left(\frac{\pi 50^3}{9}\right)^{2/3} = -2t$	**M1**		Substitute correct limits into integrals of forms $aV^{2/3}$ & bt OR substitute t=0 & $V = \frac{\pi 50^3}{9}$ to find c and substitute $V = 0$
Time $= \frac{625\pi}{3}$ secs or 654 secs (3 sf) oe	**A1**		Allow without secs or 10.9 mins or 10 mins 54 secs or: SC. Use of $r = h\sin 30$ (answer $\frac{625\pi}{4}$ or 491) can score all 4 M-marks and final A1

7.

(a)	25	B1	3.4
		(1)	
(b)	Attempts to differentiate using the product rule $\frac{dv}{dt} = \ln(t+1) \times -0.4 + \frac{(10-0.4t)}{t+1}$	M1 A1	3.1b 1.1b
	Sets their $\frac{dv}{dt} = 0 \Rightarrow \frac{(10-0.4\boldsymbol{t})}{(\boldsymbol{t}+1)} = 0.4\ln(t+1)$ **and** then makes progress towards making "$\boldsymbol{t}$" the subject (See notes for this)	dM1	1.1b
	$t = \frac{25-\ln(t+1)}{1+\ln(t+1)}$ $t = \frac{26}{1+\ln(t+1)} - 1$ *	A1*	2.1
		(4)	
(c)	(i) Attempts $t_2 = \frac{26}{1+\ln 8} - 1$	M1	1.1b
	awrt 7.298	A1	1.1b
	(ii) awrt 7.33 seconds	A1	3.2a
		(3)	
			(8 marks)

8.

$dx = 2\cos\theta d\theta$	M1	1.1a	Attempt to link dx and $d\theta$	Allow sign error only
$\int\sqrt{4-x^2}dx = \int\sqrt{4-4\sin^2\theta}.2\cos\theta d\theta$	M1	3.1a	Attempt to write integrand in terms of θ	Must substitute for both function and dx Can follow M0 but do not allow just $dx = d\theta$
$= \int\sqrt{4\cos^2\theta}.2\cos\theta d\theta$ $= \int 4\cos^2\theta d\theta$	A1	1.1	Obtain correct integrand in terms of $\cos\theta$ only	Condone no $d\theta$, as long as previously seen
$= \int(2\cos 2\theta + 2)d\theta$	M1	2.1	Attempt use of double angle formula	Using $\cos 2\theta = \pm 2\cos^2\theta \pm 1$ Integrand must be of form $k\cos^2\theta$, which must have come from correct method with coefficient errors only
$= \sin 2\theta + 2\theta$	A1FT	1.1	Integrate to obtain $\sin 2\theta + 2\theta$	FT on $a\cos 2\theta + b$ only
$[\sin 2\theta + 2\theta]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} = (\sin\frac{2}{3}\pi + \frac{2}{3}\pi) - (\sin\frac{2}{6}\pi + \frac{2}{6}\pi)$ $= (\frac{1}{2}\sqrt{3} + \frac{2}{3}\pi) - (\frac{1}{2}\sqrt{3} + \frac{1}{3}\pi)$	M1	2.1	Attempt use of limits	Must be correct limits (either x or θ, as long as consistent with their integral), correct order and subtraction Allow M1 for use of limits in any integration attempt in terms of θ Allow M1 for either expressions that still involve sin, or exact equivs M0 for decimal values, even if then stated to be the same as $\frac{1}{3}\pi$ Condone eg $\frac{1}{2}\sqrt{3}$ from $\sin 120°$, but M0 if degrees used in linear term
$= \frac{1}{3}\pi$ **A.G.**	A1	2.1	Obtain given answer of $\frac{1}{3}\pi$	Must see both surd values, or an explanation as to why $\sin\frac{2}{3}\pi = \sin\frac{2}{6}\pi$
	[7]			

9.

$\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \times \frac{1}{x} \, dx$	M1	1.1b
$= \frac{x^4}{4}\ln x - \frac{x^4}{16}(+c)$	M1 A1	1.1b 1.1b
$\int_1^{e^2} x^3 \ln x \, dx = \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^{e^2} = \left(\frac{e^8}{4}\ln e^2 - \frac{e^8}{16}\right) - \left(-\frac{1^4}{16}\right)$	M1	2.1
$= \frac{7}{16}e^8 + \frac{1}{16}$	A1	1.1b
	(5)	
		(5 marks)

10.

(a)		$\int e^{3y} dy = \int 3x^2 \ln x dx$	**M1**	**3.1a**	Separate variables and attempt integration of at least one side	Allow ke^{3y}, with $k \neq 1$, as 'attempt' at integration of LHS 'Attempt' at RHS may not be use of integration by parts Allow BOD on missing integral sign / missing dy / missing dx as long as intention clear
		$\int e^{3y} dy = \frac{1}{3}e^{3y}$	**B1**	**1.1**	Correct LHS	B0 if still part of an expression also involving x
		$\int 3x^2 \ln x dx = x^3 \ln x - \int x^2 dx$	**M1**	**3.1a**	Attempt integration by parts on RHS – must have correct parts	As far as attempt at $x^3 \ln x - \int x^2 dx$, possibly with $\int \frac{1}{x} x^3 dx$ not yet simplified
		$= x^3 \ln x - \frac{1}{3}x^3 + c$	**A1**	**1.1**	Correct RHS (condone no + c)	Condone no modulus sign on ln$x$
		$\frac{1}{3}e^3 = e^3 \ln e - \frac{1}{3}e^3 + c$ so $c = -\frac{1}{3}e^3$	**M1**	**1.1a**	Attempt use of (e, 1) to find c	Used in an equation involving x, y and c, following some integration attempt of both sides As far as finding c, either exact or as a decimal M1 can be implied by sight of $-\frac{1}{3}e^3$ or –6.695… following a correct equation
		$\frac{1}{3}e^{3y} = x^3 \ln x - \frac{1}{3}x^3 - \frac{1}{3}e^3$ $e^{3y} = 3x^3 \ln x - x^3 - e^3$	**A1**	**1.1**	Obtain correct equation, in required form	Any equivalent form on the RHS, but must be $e^{3y} = \ldots$ A0 if decimal approximation for e^3
			[6]			
(b)		$e^{3y} = 3e^6\ln e^2 - e^6 - e^3$ $= 6e^6 - e^6 - e^3$ $= 5e^6 - e^3$	**M1***	**2.1**	Substitute $x = e^2$, into their integral involving ln$x$, and attempt to simplify	ln$x$ may be lnx^p if any coefficient has been taken into the ln term As far as correctly simplifying the ln term to remove ln Must be working exactly, so M0 if decimals seen before ln dealt with
		$3y = \ln(e^3(5e^3 - 1))$ $= 3 + \ln(5e^3 - 1)$	**M1 dep***	**2.1**	Introduce logs correctly, and attempt to rearrange to given form	Their equation must have two terms, or possibly more, with the terms having a common factor of e^k Attempt must go as far as splitting into the sum of two terms, with lne^k simplified to k
		$y = 1 + \frac{1}{3}\ln(5e^3 - 1)$	**A1** **[3]**	**2.1**	Obtain $y = 1 + \frac{1}{3}\ln(5e^3 - 1)$	No need to state a, b and c explicitly