Topic X8 Further calculus (Pre-TT B) [42]

1.

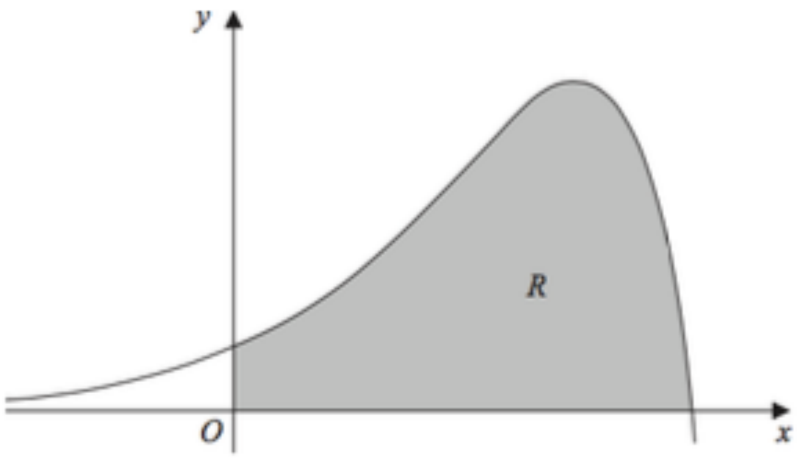


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = 2e^{2x} - xe^{2x}, \quad x \in \mathbb{R}$$

The finite region R, shown shaded in Figure 4, is bounded by the curve, the x-axis and the y-axis.

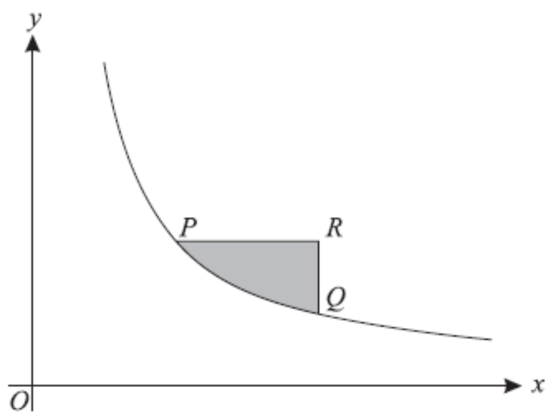
Use calculus to show that the exact area of R can be written in the form $pe^4 + q$, where p and q are rational constants to be found.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

2.

(a) Find the exact value of $\int_1^2 \frac{2}{(4x-1)^2}\,dx$. [4]

(b)



The diagram shows part of the curve $y = \frac{1}{x}$. The point P has coordinates $\left(a, \frac{1}{a}\right)$ and the point Q has coordinates $\left(2a, \frac{1}{2a}\right)$, where a is a positive constant. The point R is such that PR is parallel to the x-axis and QR is parallel to the y-axis. The region shaded in the diagram is bounded by the curve and by the lines PR and QR. Show that the area of this shaded region is $\ln\left(\frac{1}{2}e\right)$. [6]

3.

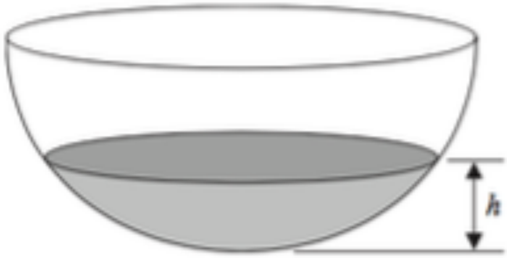


Figure 3

A bowl is modelled as a hemispherical shell as shown in Figure 3.

Initially the bowl is empty and water begins to flow into the bowl.

When the depth of the water is h cm, the volume of water, V cm³, according to the model is given by

$$V = \frac{1}{3}\pi h^2(75 - h), \qquad 0 \leqslant h \leqslant 24$$

The flow of water into the bowl is at a constant rate of $160\pi\,\text{cm}^3\text{s}^{-1}$ for $0 \leqslant h \leqslant 12$

(a) Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 10$

(5)

Given that the flow of water into the bowl is increased to a constant rate of $300\pi\,\text{cm}^3\text{s}^{-1}$ for $12 < h \leqslant 24$

(b) find the rate of change of the depth of the water, in cm s^{-1}, when $h = 20$

(2)

4.

A curve is defined by the parametric equations

$$x = \sin^2\theta, \quad y = 4\sin\theta - \sin^3\theta,$$

where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{4 - 3\sin^2\theta}{2\sin\theta}$. **[3]**

(ii) Find the coordinates of the point on the curve at which the gradient is 2. **[3]**

(iii) Show that the curve has no stationary points. **[2]**

(iv) Find a cartesian equation of the curve, giving your answer in the form $y^2 = f(x)$. **[2]**

5.

Paraffin is stored in a tank with a horizontal base. At time t minutes, the depth of paraffin in the tank is x cm. When $t = 0$, $x = 72$. There is a tap in the side of the tank through which the paraffin can flow. When the tap is opened, the flow of the paraffin is modelled by the differential equation

$$\frac{dx}{dt} = -4(x - 8)^{\frac{1}{3}}.$$

(i) How long does it take for the level of paraffin to fall from a depth of 72 cm to a depth of 35 cm? **[7]**

(ii) The tank is filled again to its original depth of 72 cm of paraffin and the tap is then opened. The paraffin flows out until it stops. How long does this take? **[3]**