Areas between Curves

Starter

1. (Review of last lesson) The surface area of a sphere, S, is expanding at a rate of 9 cm^2 s^{-1}. Find the exact rate of increase of the radius, r, when radius is 7 cm.

Working: "Find the exact rate of increase of the radius" means find $\frac{dr}{dt}$.

…surface area of a sphere is expanding at a rate of 9 cm^2 s^{-1} $\Rightarrow \frac{dS}{dt} = 9$

By the chain rule: $\frac{dr}{dt} = \frac{dS}{dt} \times \frac{dr}{dS} = \frac{\frac{dS}{dt}}{\frac{dS}{dr}}$ **so we need** $\frac{dS}{dr}$

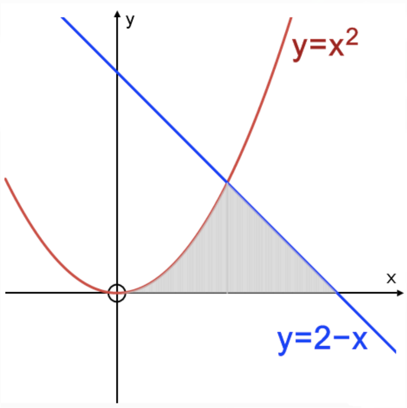
For a sphere $S = 4\pi r^2 \quad \Rightarrow \quad \frac{dS}{dr} = 8\pi r$

$\therefore \frac{dr}{dt} = \frac{\frac{dS}{dt}}{\frac{dS}{dr}} = \frac{9}{8\pi r}$

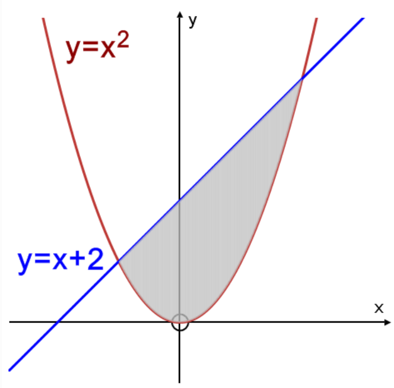
So when $r = 7$, $\frac{dr}{dt} = \frac{9}{8\pi \times 7} = \frac{9}{56\pi}$ cm s^{-1}

2. Find the area of the shaded part:

(a)



(b)



Working: (a) To find the point of intersection illustrated, solve $x^2 = 2 - x$

$\therefore x^2 + x - 2 = 0 \quad \Rightarrow \quad (x+2)(x-1) = 0$

Since $x > 0, x = 1$

Also $y = 2 - x$ intersects the $x-$axis when $y = 0 \quad \Rightarrow \quad x = 2$

$$\text{Area} = \int_0^1 x^2 dx + \int_1^2 (2-x)dx$$

$$= \left[\frac{1}{3}x^3\right]_0^1 + \left[2x - \frac{1}{2}x^2\right]_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left(\left(2 \times 2 - \frac{1}{2}x2^2\right) - \left(2 \times 1 - \frac{1}{2}x2^2\right)\right)$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

(b) To find the points of intersection, solve $x^2 = x + 2$

$\therefore x^2 - x - 2 = 0 \quad \Rightarrow \quad (x-2)(x+1) = 0$

$x = -1$ and $x = 2$

The line $y = x + 2$ is above the curve $y = x^2$. The required area is ***the area under the curve subtracted from the area under the line***.

$$\therefore \text{Area} = \int_{-1}^{2} (x+2)dx - \int_{-1}^{2} x^2 dx$$

$$= \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^{2}$$

N.B. The integrals can be combined since the limits are the same.

$$= \left(\frac{1}{2} \times 2^2 + 2 \times 2 - \frac{1}{3} \times 2^3\right) - \left(\frac{1}{2} \times (-1)^2 + 2 \times (-1) - \frac{1}{3}(-1)^3\right)$$

Shaded area $= 4.5$

E.g. 1 Find the area between:

(a) the curve $y = x^3$ and the line $y = 4x$

(b) the curves $y = \sin x + 1$ and $y = \cos x + 1$ for $0 \le x \le 2\pi$.

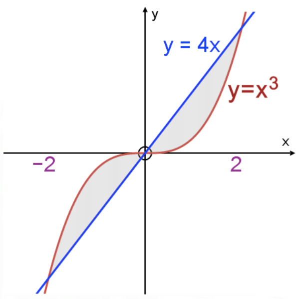
Working: (a) To find points of intersection, solve $x^3 = 4x$

$\therefore x^3 - 4x = 0 \quad \Rightarrow \quad x(x^2 - 4) = 0$

$x(x-2)(x+2) = 0$

$\Rightarrow \quad x = -2, x = 0$ and $x = 2$

The line $y = 4x$ is above the curve $y = x^3$.
Since areas are symmetrical, find the area from $x = 0$ to $x = 2$ and then double it.



$$\therefore \text{Area} = 2\left(\int_{0}^{2} 4x dx - \int_{0}^{2} x^3 dx\right)$$

$$= 2\left[2x^2 - \frac{1}{4}x^4\right]_{0}^{2}$$

N.B. The integrals can be combined since the limits are the same.

$$= 2\left(2 \times 2^2 - \frac{1}{4} \times 2^4 - 0\right)$$

Area between the curves $= 8$

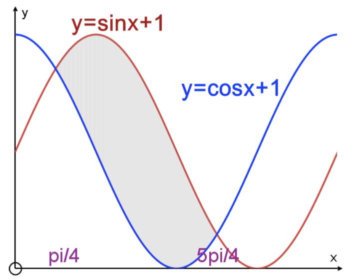
(b) To find points of intersection, solve

$\sin x + 1 = \cos x + 1$

$\therefore \sin x = \cos x \Rightarrow \tan x = 1$

$x = \dfrac{\pi}{4}$ or $x = \dfrac{5\pi}{4}$

$y = \sin x + 1$ is above $y = \cos x + 1$.



$$\therefore \text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x + 1)dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\cos x + 1)dx$$

$$= \Big[(-\cos x + x) - (\sin x + x)\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

N.B. The integrals can be combined since the limits are the same.

$$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \Big[\cos x + \sin x\Big]_{\frac{5\pi}{4}}^{\frac{\pi}{4}}$$ ***swap limits to change sign of integral***

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

Area between the curves $= 2\sqrt{2}$

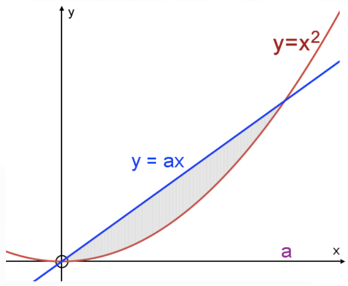
E.g. 2 The area between the graphs $y = x^2$ and $y = ax$ is 36, where a is a constant and $a > 0$. Find the value of a.

Working: To find points of intersection, solve $x^2 = ax$

$\therefore x^2 - ax = 0 \Rightarrow x(x-a) = 0$

$\Rightarrow x = 0$ and $x = a$

The line $y = ax$ is above the curve $y = x^2$.



$$\therefore \text{Area} = \int_0^a ax\,dx - \int_0^a x^2 dx$$

$$= \left[\frac{a}{2}x^2 - \frac{1}{3}x^3\right]_0^a$$

N.B. The integrals can be combined since the limits are the same.

$$= \left(\frac{1}{2}a^3 - \frac{1}{3}a^3\right) - (0) = \frac{1}{6}a^3$$

Since the area is 36: $\dfrac{1}{6}a^3 = 36 \Rightarrow a^3 = 216$

$\therefore a = 6$

Video A: **Area between 2 curves**
Video B: **Area between 2 curves**

Solutions to Starter and E.g.s

Exercise
p274 12F Qu 1i, 3-6, 9, 11 (10)