Binomial expansion of compound expressions

Starter

1. **(Review of last lesson)** Find the first ***three*** terms in the expansion of $\dfrac{1}{\sqrt[3]{8-5x}}$ and state the range of values for which it is valid.

Working: $\dfrac{1}{\sqrt[3]{8-5x}} = \dfrac{1}{\sqrt[3]{8\left(1-\frac{5x}{8}\right)}} = \dfrac{1}{\sqrt[3]{8}\left(1-\frac{5x}{8}\right)^{\frac{1}{3}}} = \dfrac{\left(1-\frac{5x}{8}\right)^{-\frac{1}{3}}}{\sqrt[3]{8}} = \dfrac{1}{2}\left(1-\dfrac{5x}{8}\right)^{-\frac{1}{3}}$

$$\frac{1}{2}\left(1-\frac{5x}{8}\right)^{-\frac{1}{3}} = \frac{1}{2}\left(1-\frac{1}{3}\left(-\frac{5x}{8}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{1\times 2}\left(-\frac{5x}{8}\right)^2+\ldots\right)$$
$$= \frac{1}{2}\left(1+\frac{5}{24}x+\frac{25}{288}x^2+\ldots\right)$$
$$= \frac{1}{2}+\frac{5}{48}x+\frac{25}{576}x^2+\ldots$$

The expansion is valid for $-1 < -\dfrac{5}{8}x < 1$ i.e. $\dfrac{8}{5} > x > -\dfrac{8}{5}$

$\therefore -\dfrac{8}{5} < x < \dfrac{8}{5}$

2. Given that the coefficient of x^3 in the expansion of $\dfrac{1}{(1+ax)^3}$ is -2160, find a.

Working: $\dfrac{1}{(1+ax)^3} = (1+ax)^{-3}$

The term in x^3 is $\dfrac{(-3)(-4)(-5)}{1\times 2\times 3}(ax)^3 = -10a^3x^3$

$\therefore -10a^3 = -2160 \quad \Rightarrow \quad a^3 = 216 \quad \Rightarrow \quad a = 6$

E.g. 1 Find in ascending powers of x, up to and including the x^2 term, the expansion of $\dfrac{x+1}{\sqrt{1-x}}$, stating the values of x for which the expansion is valid.

Working: $\dfrac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$

$$(1-x)^{-\frac{1}{2}} = 1-\frac{1}{2}(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2}(-x)^2+\ldots$$
$$= 1+\frac{1}{2}x+\frac{3}{8}x^2+\ldots$$
$$\frac{x+1}{\sqrt{1-x}} = \left(x+1\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\ldots\right)$$
$$= x+\frac{1}{2}x^2+1+\frac{1}{2}x+\frac{3}{8}x^2+\ldots$$ ***ignoring terms above x^2***
$$= 1+\frac{3}{2}x+\frac{7}{8}x^2+\ldots$$

The expansion is valid for $-1 < x < 1$

E.g. 2 (a) Find the first three terms in the expansion of $\dfrac{\sqrt{1+2x}}{\sqrt{1-4x}}$ in ascending powers of x.

(b) State the values of x for which the expansion is valid.

(c) By substituting $x = 0.01$ in your expansion, find an approximation for $\sqrt{17}$.

Working: (a)

$$(1+2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{1\times 2}(2x)^2 + \dots$$
$$= 1 + x - \frac{1}{2}x^2 + \dots$$

$$(1-4x)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2}(-4x)^2 + \dots$$
$$= 1 + 2x + 6x^2 + \dots$$

$$\frac{\sqrt{1+2x}}{\sqrt{1-4x}} = \left(1 + x - \frac{1}{2}x^2 + \dots\right)\left(1 + 2x + 6x^2 + \dots\right)$$
$$= 1 + 2x + 6x^2 + x + 2x^2 - \frac{1}{2}x^2 + \dots$$
$$= 1 + 3x + \frac{15}{2}x^2 + \dots$$

(b) $(1+2x)^{\frac{1}{2}}$ is valid for $-1 < 2x < 1$ i.e. $-\frac{1}{2} < x < \frac{1}{2}$

$(1-4x)^{-\frac{1}{2}}$ is valid for $-1 < (-4x) < 1$ i.e. $-\frac{1}{4} < x < \frac{1}{4}$

So the expansion of $\dfrac{\sqrt{1+2x}}{\sqrt{1-4x}}$ is valid for $-\frac{1}{4} < x < \frac{1}{4}$

N.B. **Choose the narrowest range of values.**

(c) When $x = 0.01$,

$$\frac{\sqrt{1+2x}}{\sqrt{1-4x}} = \frac{\sqrt{1+2\times 0.01}}{\sqrt{1-4\times 0.01}} = \frac{\sqrt{1.02}}{\sqrt{0.96}} = \sqrt{\frac{17}{16}} = \frac{\sqrt{17}}{4}$$

$\sqrt{17} = 4 \times \dfrac{\sqrt{1+2x}}{\sqrt{1-4x}}$ when $x = 0.01$

Using the expansion from (a):

$$\sqrt{17} \approx 1 + 3\times 0.01 + \frac{15}{2}\times 0.01^2$$
$$= 4(1 + 0.03 + 0.00075)$$
$$= 4.123$$

Exercise

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E.g. 3 Find the first three terms in ascending powers of x in the expansion of $\dfrac{5x+7}{(x+1)(x+2)}$, stating the values of x for which this is valid.

Working:

$$\frac{5x+7}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$

Multiply by $(x+1)(x+2)$:

$$5x+7 \equiv A(x+2) + B(x+1)$$

Let $x = -2$: $\quad -3 = -B \quad \therefore B = 3$

Let $x = -1$: $\quad 2 = A$

So $\dfrac{5x+7}{(x+1)(x+2)} \equiv \dfrac{2}{x+1} + \dfrac{3}{x+2}$

$$\frac{5x+7}{(x+1)(x+2)} \equiv \frac{2}{1+x} + \frac{3}{2+x}$$

$$\equiv \frac{2}{1+x} + \frac{3}{2\left(1+\frac{x}{2}\right)}$$

$$\equiv 2(1+x)^{-1} + \frac{3}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$$2(1+x)^{-1} = 2\left(1 - x + \frac{(-1)(-2)}{1\times 2}x^2 + \ldots\right)$$

$$= 2\left(1 - x + x^2 + \ldots\right)$$

$$\frac{3}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{3}{2}\left(1 - 1\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{1\times 2}\left(\frac{x}{2}\right)^2 + \ldots\right)$$

$$= \frac{3}{2}\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right)$$

$$\frac{5x+7}{(x+1)(x+2)} = 2\left(1 - x + x^2 + \ldots\right) + \frac{3}{2}\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right)$$

$$= \frac{7}{2} - \frac{11}{4}x + \frac{19}{8}x^2 + \ldots$$

$2(1+x)^{-1}$ is valid for $-1 < x < 1$

$\frac{3}{2}\left(1+\frac{x}{2}\right)^{-1}$ is valid for $-1 < \frac{x}{2} < 1$ i.e. $-2 < x < 2$

Therefore, the expansion of $\dfrac{5x+7}{(x+1)(x+2)}$ is valid for $-1 < x < 1$.

E.g. 4 By express $\dfrac{x+7}{(x-1)(x+3)}$ as partial fractions, find its expansion in ascending powers of x up until the term in x^2.

Working: $\dfrac{x+7}{(x-1)(x+3)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+3}$

Multiply by $(x-1)(x+3)$***:***

$$x+7 \equiv A(x+3) + B(x-1)$$

Let $x = -3$: $\quad 4 = -4B \quad \therefore B = -1$

Let $x = 1$: $\quad 8 = 4A \quad \therefore A = 2$

So $\dfrac{x+7}{(x-1)(x+3)} \equiv \dfrac{2}{x-1} - \dfrac{1}{x+3}$

$$\frac{x+7}{(x-1)(x+3)} \equiv -\frac{2}{1-x} - \frac{1}{3+x}$$

$$\equiv -\frac{2}{1-x} - \frac{1}{3\left(1+\frac{x}{3}\right)}$$

$$\equiv -2(1-x)^{-1} - \frac{1}{3}\left(1+\frac{x}{3}\right)^{-1}$$

$$-2(1-x)^{-1} = -2\left(1-(-x)+\frac{(-1)(-2)}{1\times 2}x^2+\ldots\right)$$

$$= -2\left(1+x+x^2+\ldots\right)$$

$$-\frac{1}{3}\left(1+\frac{x}{3}\right)^{-1} = -\frac{1}{3}\left(1-1\left(\frac{x}{3}\right)+\frac{(-1)(-2)}{1\times 2}\left(\frac{x}{3}\right)^2+\ldots\right)$$

$$= -\frac{1}{3}\left(1-\frac{1}{3}x+\frac{1}{9}x^2+\ldots\right)$$

$$\frac{x+7}{(x-1)(x+3)} = -2\left(1+x+x^2+\ldots\right) - \frac{1}{3}\left(1-\frac{1}{3}x+\frac{1}{9}x^2+\ldots\right)$$

$$= -\frac{7}{3} - \frac{17}{9}x - \frac{55}{27}x^2 + \ldots$$

$-2(1-x)^{-1}$ is valid for $-1 < (-x) < 1$ i.e. $-1 < x < 1$

$-\dfrac{1}{3}\left(1+\dfrac{x}{3}\right)^{-1}$ is valid for $-1 < \dfrac{x}{3} < 1$ i.e. $-3 < x < 3$

Therefore, the expansion of $\dfrac{x+7}{(x-1)(x+3)}$ is valid for $-1 < x < 1$.

Video: General binomial expansion (from 1:19:51)

Partial fractions and the binomial expansion EQ

Solutions to Starter and E.g.s

Exercise

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