

## Calculus with Vectors

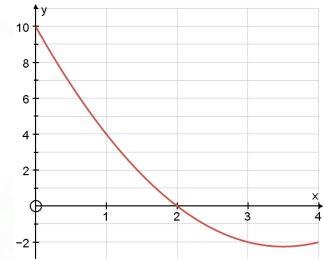
### Starter

1. A particle's velocity is given by  $v = t^2 - 7t + 10$  for  $0 \leq t \leq 4$ . Find:
- its acceleration when  $t = 3$ .
  - the displacement for  $0 \leq t \leq 4$  and
  - the total distance travelled for  $0 \leq t \leq 4$ .

**Working:** (a)  $v = t^2 - 7t + 10 \Rightarrow a = \frac{dv}{dt} = 2t - 7$   
 When  $t = 3$ ,  $a = 2 \times 3 - 7 = -1$

(b) Displacement =  $\int_0^4 (t^2 - 7t + 10)dt$   
 $= \left[ \frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t \right]_0^4$   
 $= \left( \frac{1}{3} \times 4^3 - \frac{7}{2} \times 4^2 + 10 \times 4 \right) - 0$   
 $= 5\frac{1}{3}$

- (c) Total distance travelled requires the areas above and below the  $x$ -axis to be calculated separately  
 Solving  $t^2 - 7t + 10 = 0$ :  
 $(t - 2)(t - 5) = 0$   
 $\therefore t = 2$  is where the area changes from positive to negative.



Distance travelled  $0 \leq t \leq 2 = \int_0^2 (t^2 - 7t + 10)dt$   
 $= \left[ \frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t \right]_0^2$   
 $= \left( \frac{1}{3} \times 2^3 - \frac{7}{2} \times 2^2 + 10 \times 2 \right) - 0$   
 $= 8\frac{2}{3}$

Distance travelled  $2 \leq t \leq 4 = \int_2^4 (t^2 - 7t + 10)dt$   
 $= \left[ \frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t \right]_2^4$   
 $= \left( \frac{1}{3} \times 4^3 - \frac{7}{2} \times 4^2 + 10 \times 4 \right) - \left( \frac{1}{3} \times 2^3 - \frac{7}{2} \times 2^2 + 10 \times 2 \right)$   
 $= 5\frac{1}{3} - 8\frac{2}{3} = -3\frac{1}{3}$

This area needs to be made positive.

So distance travelled =  $8\frac{2}{3} + 3\frac{1}{3} = 12$  m

2. **(Review of A2 material)** The path of a particle is given by the parametric equations  $x = t + 3$ ,  $y = 2t^2 - 5t + 8$ . Find the cartesian equation of the path.

**Working:**  $x = t + 3 \Rightarrow t = x - 3$   
 Replace  $t$  by  $x - 3$  in  $y = 2t^2 - 5t + 8$ :  $y = 2(x - 3)^2 - 5(x - 3) + 8$   
 $y = 2(x^2 - 6x + 9) - 5x + 15 + 8 \Rightarrow y = 2x^2 - 17x + 41$

**E.g. 1** A particle is moving in a vertical plane so that at time  $t$  seconds it has velocity  $v$  m/s where  $\mathbf{v} = (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}$ . When  $t = 2$ , the particle has position vector  $(10\mathbf{i} + 3\mathbf{j})$  m with respect to a fixed origin O.

- (a) Find the acceleration,  $a$ , of the particle at time  $t$ .  
 (b) Find the position vector of the particle when  $t = 4$ .  
 (c) Find the value of  $t$  for which the particle is directly above O.  
**N.B.** "Directly above O" means the  $\mathbf{i}$  component is zero.

**Working:** (a)  $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{i} + (3t^2 - 6)\mathbf{j}$  m/s<sup>2</sup>.

(b)  $\mathbf{s} = \int (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j} dt$   
 $\mathbf{s} = (8t + t^2)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2)\mathbf{j} + \mathbf{c}$   
 When  $t = 2$ ,  $\mathbf{s} = (10\mathbf{i} + 3\mathbf{j})$   
 $10\mathbf{i} + 3\mathbf{j} = (8 \times 2 + 2^2)\mathbf{i} + (\frac{1}{4} \times 2^4 - 3 \times 2^2)\mathbf{j} + \mathbf{c}$   
 $\mathbf{c} = -10\mathbf{i} + 11\mathbf{j}$   
 $\therefore \mathbf{s} = (t^2 + 8t - 10)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2 + 11)\mathbf{j}$   
 When  $t = 4$ ,  $\mathbf{s} = (4^2 + 8 \times 4 - 20)\mathbf{i} + (\frac{1}{4} \times 4^4 - 3 \times 4^2 + 11)\mathbf{j}$   
 The position vector when  $t = 4$  is  $38\mathbf{i} + 27\mathbf{j}$ .

(c) Solve  $t^2 + 8t - 10 = 0$   
 Since  $t > 0$ ,  $t = -4 + \sqrt{26} = 1.10$  s (3 s.f.)

**E.g. 2** A body of mass 3 kg moves under the action of a force,  $\mathbf{F}$ , where  $\mathbf{F} = (2t + 1)\mathbf{i} + (11 - 6t^2)\mathbf{j}$  m/s<sup>2</sup>. Initially the body has velocity  $(4\mathbf{i} + 9\mathbf{j})$  m/s.

(a) Find the velocity of the body after 2 seconds.

(b) Find the speed and angle of direction of the body after 5 seconds.

**N.B.** For (b), the direction of motion is given by the velocity and its angle is the angle it makes with the positive  $x$ -axis.

**Working:** (a) Using  $\mathbf{F} = m\mathbf{a}$ :  $\mathbf{a} = \frac{2t + 1}{3}\mathbf{i} + \frac{11 - 6t^2}{3}\mathbf{j}$

$$\mathbf{v} = \int \frac{2t + 1}{3}\mathbf{i} + \frac{11 - 6t^2}{3}\mathbf{j} dt$$

$$\mathbf{v} = \left(\frac{t^2}{3} + \frac{1}{3}t\right)\mathbf{i} + \left(\frac{11t}{3} - \frac{2}{3}t^3\right)\mathbf{j} + \mathbf{c}$$

When  $t = 0$ ,  $\mathbf{v} = 4\mathbf{i} + 9\mathbf{j} \Rightarrow \mathbf{c} = 4\mathbf{i} + 9\mathbf{j}$

$$\mathbf{v} = \left(\frac{t^2}{3} + \frac{1}{3}t + 4\right)\mathbf{i} + \left(\frac{11t}{3} - \frac{2}{3}t^3 + 9\right)\mathbf{j}$$

When  $t = 2$ ,  $\mathbf{v} = \left(\frac{2^2}{3} + \frac{2}{3} + 4\right)\mathbf{i} + \left(\frac{22}{3} - \frac{2}{3} \times 2^3 + 9\right)\mathbf{j}$

$\therefore \mathbf{v} = 6\mathbf{i} + 11\mathbf{j}$

(b) When  $t = 5$ ,  $\mathbf{v} = \left(\frac{5^2}{3} + \frac{5}{3} + 4\right)\mathbf{i} + \left(\frac{55}{3} - \frac{2}{3} \times 5^3 + 9\right)\mathbf{j}$

$\mathbf{v} = 14\mathbf{i} - 56\mathbf{j}$  *this is the velocity*

Speed =  $|\mathbf{v}| = |14\mathbf{i} - 56\mathbf{j}| = 57.7$  (3 s.f.)

Direction is the angle the velocity vector makes with the +ve  $x$ -axis

Direction =  $360^\circ - \tan^{-1} \frac{56}{14} = 284^\circ$  (3 s.f.)

**Video:** [Variable acceleration with vectors](#)

**Video:** [Calculus in 2-D kinematics example](#)

[Solutions to Starter and E.g.s](#)

### Exercise

p438 19C Qu 1i, 2i, 3i, 4-11