Calculus with Vectors

Starter

1. A particle's velocity is given by $v = t^2 - 7t + 10$ for $0 \le t \le 4$. Find:
   (a) its acceleration when $t = 3$.
   (b) the displacement for $0 \le t \le 4$ and
   (c) the total distance travelled for $0 \le t \le 4$.

Working: (a) $v = t^2 - 7t + 10 \quad \Rightarrow \quad a = \dfrac{dv}{dt} = 2t - 7$

When $t = 3$, $a = 2 \times 3 - 7 = -1$

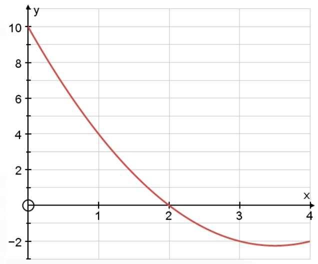
(b) Displacement $= \displaystyle\int_0^4 (t^2 - 7t + 10)dt$

$$= \left[\frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t\right]_0^4$$

$$= \left(\frac{1}{3} \times 4^3 - \frac{7}{2} \times 4^2 + 10 \times 4\right) - 0$$

$$= 5\frac{1}{3}$$

(c) Total distance travelled requires the areas above and below the $x-$axis to be calculated separately



Solving $t^2 - 7t + 10 = 0$:

$(t - 2)(t - 5) = 0$

$\therefore t = 2$ is where the area changes from positive to negative.

Distance travelled $0 \le t \le 2 = \displaystyle\int_0^2 (t^2 - 7t + 10)dt$

$$= \left[\frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t\right]_0^2$$

$$= \left(\frac{1}{3} \times 2^3 - \frac{7}{2} \times 2^2 + 10 \times 2\right) - 0$$

$$= 8\frac{2}{3}$$

Distance travelled $2 \le t \le 4 = \displaystyle\int_2^4 (t^2 - 7t + 10)dt$

$$= \left[\frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t\right]_2^4$$

$$= \left(\frac{1}{3} \times 4^3 - \frac{7}{2} \times 4^2 + 10 \times 4\right) - \left(\frac{1}{3} \times 2^3 - \frac{7}{2} \times 2^2 + 10 \times 2\right)$$

$$= 5\frac{1}{3} - 8\frac{2}{3} = -3\frac{1}{3}$$

This area needs to be made positive.

So distance travelled $= 8\dfrac{2}{3} + 3\dfrac{1}{3} = 12$ m

2. **(Review of A2 material)** The path of a particle is given by the parametric equations $x = t + 3$, $y = 2t^2 - 5t + 8$. Find the cartesian equation of the path.

Working: $x = t + 3 \quad \Rightarrow \quad t = x - 3$

Replace t by $x - 3$ in $y = 2t^2 - 5t + 8$: $\quad y = 2(x-3)^2 - 5(x-3) + 8$

$y = 2(x^2 - 6x + 9) - 5x + 15 + 8 \quad \Rightarrow \quad y = 2x^2 - 17x + 41$

E.g. 1 A particle is moving in a vertical plane so that at time t seconds it has velocity v m/s where $\mathbf{v} = (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}$. When $t = 2$, the particle has position vector $(10\mathbf{i} + 3\mathbf{j})$ m with respect to a fixed origin O.

(a) Find the acceleration, $\mathbf{a}$, of the particle at time t.

(b) Find the position vector of the particle when $t = 4$.

(c) Find the value of t for which the particle is directly above O.

N.B. "Directly above O" means the $\mathbf{i}$ component is zero.

Working: (a) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2\mathbf{i} + (3t^2 - 6)\mathbf{j}$ m/s².

(b) $\mathbf{s} = \displaystyle\int (8 + 2t)\mathbf{i} + (t^3 - 6t)\mathbf{j}\,dt$

$\mathbf{s} = (8t + t^2)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2)\mathbf{j} + \mathbf{c}$

When $t = 2$, $\mathbf{s} = (10\mathbf{i} + 3\mathbf{j})$

$10\mathbf{i} + 3\mathbf{j} = (8 \times 2 + 2^2)\mathbf{i} + (\frac{1}{4} \times 2^4 - 3 \times 2^2)\mathbf{j} + \mathbf{c}$

$\mathbf{c} = -10\mathbf{i} + 11\mathbf{j}$

$\therefore \mathbf{s} = (t^2 + 8t - 10)\mathbf{i} + (\frac{1}{4}t^4 - 3t^2 + 11)\mathbf{j}$

When $t = 4$, $\mathbf{s} = (4^2 + 8 \times 4 - 20)\mathbf{i} + (\frac{1}{4} \times 4^4 - 3 \times 4^2 + 11)\mathbf{j}$

The position vector when $t = 4$ is $38\mathbf{i} + 27\mathbf{j}$.

(c) Solve $t^2 + 8t - 10 = 0$

Since $t > 0$, $t = -4 + \sqrt{26} = 1.10$ s (3 s.f.)

E.g. 2 A body of mass 3 kg moves under the action of a force, $\mathbf{F}$, where $\mathbf{F} = (2t+1)\mathbf{i} + (11-6t^2)\mathbf{j}$ N. Initially the body has velocity $(4\mathbf{i} + 9\mathbf{j})$ m/s.

(a) Find the velocity of the body after 2 seconds.

(b) Find the speed and angle of direction of the body after 5 seconds.

N.B. For (b), the direction of motion is given by the velocity and its angle is is the angle it makes with the positive x-axis.

Working: (a) Using $\mathbf{F} = m\mathbf{a}$: $\mathbf{a} = \frac{2t+1}{3}\mathbf{i} + \frac{11-6t^2}{3}\mathbf{j}$

$$\mathbf{v} = \int \frac{2t+1}{3}\mathbf{i} + \frac{11-6t^2}{3}\mathbf{j}\,dt$$

$$\mathbf{v} = \left(\frac{t^2}{3} + \frac{1}{3}t\right)\mathbf{i} + \left(\frac{11t}{3} - \frac{2}{3}t^3\right)\mathbf{j} + \mathbf{c}$$

When $t = 0$, $\mathbf{v} = 4\mathbf{i} + 9\mathbf{j} \quad \Rightarrow \quad \mathbf{c} = 4\mathbf{i} + 9\mathbf{j}$

$$\mathbf{v} = \left(\frac{t^2}{3} + \frac{1}{3}t + 4\right)\mathbf{i} + \left(\frac{11t}{3} - \frac{2}{3}t^3 + 9\right)\mathbf{j}$$

When $t = 2, \mathbf{v} = \left(\frac{2^2}{3} + \frac{2}{3} + 4\right)\mathbf{i} + \left(\frac{22}{3} - \frac{2}{3} \times 2^3 + 9\right)\mathbf{j}$

$\therefore \mathbf{v} = 6\mathbf{i} + 11\mathbf{j}$

(b) When $t = 5$, $\mathbf{v} = \left(\frac{5^2}{3} + \frac{5}{3} + 4\right)\mathbf{i} + \left(\frac{55}{3} - \frac{2}{3} \times 5^3 + 9\right)\mathbf{j}$

$\mathbf{v} = 14\mathbf{i} - 56\mathbf{j}$ ***this is the velocity***

Speed $= |\mathbf{v}| = \left|14\mathbf{i} - 56\mathbf{j}\right| = 57.7$ (3 s.f.)

Direction is the angle the velocity vector makes with the +ve $x-$axis

Direction $= 360^{\circ} - \tan^{-1}\frac{56}{14} = 284^{\circ}$ (3 s.f.)

Video: Variable acceleration with vectors
Video: Calculus in 2-D kinematics example

Solutions to Starter and E.g.s

Exercise
p438 19C Qu 1i, 2i, 3i, 4-11