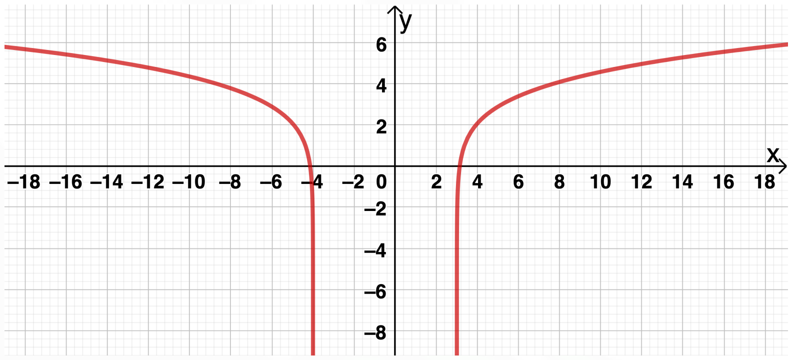
Composite functions

Starter

1. For the function, $y = \ln(x^2 + x - 12)$, find the largest possible domain and the corresponding range. Write your answers in set notation.

Working: $\ln f(x)$ ***exists when*** $f(x) > 0$.
i.e. $x^2 + x - 12 > 0 \quad \Rightarrow \quad (x-3)(x+4) > 0$
Critical values are $x = 3$ and $x = -4$.
$> 0 \quad \Rightarrow$ above the $x-$axis so $x < -4$ and $x > 3$.
Hence, the domain is $\{x : x < -4\} \cup \{x : x > 3\}$.
The range is $y \in \mathbb{R}$.



2. Let $f(x) = x^2$ and $g(x) = 5x - 3$. Find:
(a) $f(-7)$ (b) $g(6)$ (c) $f(x-5)$ (d) $g(2x-9)$

Working:
(a) $f(-7) = (-7)^2 = 49$

(b) $g(6) = 5 \times 6 - 3 = 27$

(c) $f(x-5) = (x-5)^2 = x^2 - 10x + 25$

(d) $g(2x-9) = 5 \times (2x-9) - 3 = 10x - 48$

E.g. Let $f(x) = 3x - 1$ and $g(x) = x^2 - 6$.
(a) Find the value of: (i) $fg(-4)$ (ii) $gf(2)$
(b) Find expressions for: (i) $gf(x)$ (ii) $fg(x)$

Working:
(a) (i) $fg(-4) \quad \Rightarrow \quad$ do g first
$g(-4) = (-4)^2 - 6 = 10$
$fg(-4) = f(10) = 3 \times 10 - 1 = 29$

(ii) $gf(2) \quad \Rightarrow \quad$ do f first
$f(2) = 3 \times 2 - 1 = 5$
$gf(2) = g(5) = 5^2 - 6 = 19$

(b) (i) $gf(x)$ substitutes $f(x)$ into $g(x)$
$gf(x) = g(3x-1) = (3x-1)^2 - 6 = 9x^2 - 6x - 5$

(ii) $fg(x)$ substitutes $g(x)$ into $f(x)$
$fg(x) = f(x^2 - 6) = 3(x^2 - 6) - 1 = 3x^2 - 19$

E.g. 1 Let $f(x) = 8x - 15$, $g(x) = x^2 + 1$ and $h(x) = \frac{1}{x}$, where $x \in \mathbb{R}$ and $x \neq 0$ or $x \neq \frac{15}{8}$.

(a) Find the exact value of:

(i) $fg(-3)$ (ii) $gf(3)$ (iii) $fhg(4)$ (iv) $hgf(1)$

(b) Find expressions in terms of x for:

(i) $hf(x)$ (ii) $fh(x)$ (iii) $ff(x)$
(iv) $g^2(x)$ (v) $fgf(x)$ (vi) $hgf(x)$

Working: (a) (i)

$$g(-3) = (-3)^2 + 1 = 10$$
$$fg(-3) = f(10) = 8 \times 10 - 15 = 65$$

(ii)

$$f(3) = 8 \times 3 - 15 = 9$$
$$gf(3) = g(9) = 9^2 + 1 = 82$$

(iii)

$$g(4) = 4^2 + 1 = 17$$
$$h(17) = \frac{1}{17}$$
$$fhg(4) = f\left(\frac{1}{17}\right) = 8 \times \frac{1}{17} - 15 = -\frac{247}{17} = -14\frac{9}{17}$$

(iv)

$$hgf(1) = hg(-7) = h(50) = \frac{1}{50}$$

(b) (i)

$$hf(x) = h(8x - 15) = \frac{1}{8x - 15}$$

(ii)

$$fh(x) = f\left(\frac{1}{x}\right) = \frac{8}{x} - 15$$

(iii)

$$ff(x) = f(8x - 15) = 8(8x - 15) - 15 = 64x - 135$$

(iv)

$$g^2(x) = g(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2$$

(v)

$$\begin{aligned} fgf(x) &= fg(8x - 15) \\ &= f(64x^2 - 240x + 226) \\ &= 8(64x^2 - 240x + 226) - 15 \\ &= 512x^2 - 1920x + 1793 \end{aligned}$$

(vi)

$$\begin{aligned} hgf(x) &= hg(8x - 15) \\ &= h(64x^2 - 240x + 225 + 1) \\ &= \frac{1}{64x^2 - 240x + 226} \end{aligned}$$

E.g. 2 Given that $f(x) = ax + b$ and $f(f(x)) = 9x - 28$, find the possible values of a and b.

Working: $f(x) = ax + b \Rightarrow ff(x) = f(ax + b) = a(ax + b) + b$

$\therefore a^2x + ab + b = 9x - 28$

Equating coefficients:

x: $a^2 = 9 \Rightarrow a = \pm 3$

constant: $ab + b = -28$

When $a = 3$: $3b + b = -28 \Rightarrow b = -7$

When $a = -3$: $-3b + b = -28 \Rightarrow b = 14$

E.g. 3 The functions $f(x) = 4x + 1$ and $g(x) = ax + b$ are such that $fg = gf$ for $x \in \mathbb{R}$. Find an expression for a in terms of b.

Working: $fg(x) = f(ax + b) = 4(ax + b) + 1 = 4ax + 4b + 1$

$gf(x) = g(4x + 1) = a(4x + 1) + b = 4ax + a + b$

Equating the constant terms: $4b + 1 = a + b$

$\therefore a = 3b + 1$

Video: **Composite functions**

Solutions to Starter and E.g.s

Exercise

p24 2C Qu 1i, 2i, 3i, 4, 5i, 6-8, (9 red)