Compound angle identities

Starter

1. Find the perimeter of a segment cut off by a chord of length 14 cm from a circle of radius 25 cm.

Working: *By drawing a dotted line, two right-angled triangles are created.*

$\sin P\hat{O}X = \frac{7}{25} \quad \Rightarrow \quad P\hat{O}X \approx 0.284^c$

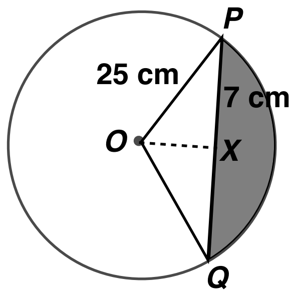
So $P\hat{O}Q \approx 2 \times 0.284^c \approx 0.5676^c$

Perimeter of segment = Chord length + $r\theta^c$

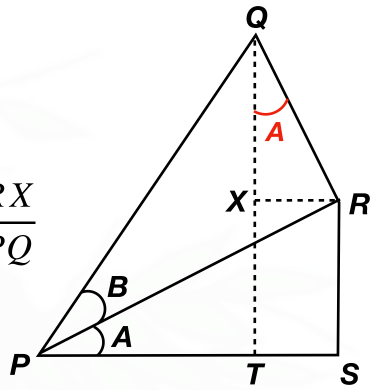
$= 14 + 25 \times 0.5676^c$

≈ 28.2

The perimeter of the segment is 28.2 cm² (3 s.f.).



E.g. 1* Using a similar method as above, derive a formula for $\cos(A+B)$ in terms $\cos A$, $\sin A$, $\cos B$ and $\sin B$.



Working: $\cos(A+B) = \frac{PT}{PQ}$

But $PT = PS - ST$:

$\cos(A+B) = \frac{PS - ST}{PQ} = \frac{PS}{PQ} - \frac{ST}{PQ} = \frac{PS}{PQ} - \frac{RX}{PQ}$

PS and PQ are not in the same triangle but we can connect them using PR.

Similarly RX and PQ can be connected via QR

$\cos(A+B) = \frac{PS}{PQ} \times \frac{PR}{PR} - \frac{RX}{PQ} \times \frac{QR}{QR}$

$\cos(A+B) = \frac{PS}{PR} \times \frac{PR}{PQ} - \frac{RX}{QR} \times \frac{QR}{PQ}$

Using trigonometry: $\frac{PS}{PR} = \cos A \qquad \frac{PR}{PQ} = \cos B$

$\frac{RX}{QR} = \sin A \qquad \frac{QR}{PQ} = \sin B$

$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$

E.g. 2 Using the facts that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, write down formulae for:

(a) $\sin(A - B)$ (b) $\cos(A - B)$

Working:

(a) $\sin(A-B) = \sin(A + (-B))$

$= \sin A \cos(-B) + \cos A \sin(-B)$

$= \sin A \cos B - \cos A \sin B$

(b) $\cos(A-B) = \cos(A + (-B))$

$= \cos A \cos(-B) - \sin A \sin(-B)$

$= \cos A \cos B + \sin A \sin B$

E.g. 3 Express $\tan(A-B)$ in terms of $\tan A$ and $\tan B$.

Working: Since $\tan(-x) = -\tan x$:
$$\tan(A-B) = \tan(A+(-B))$$
$$= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$
$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

E.g. 4 Given that we know the exact value for sine, cosine and tangent of $30°$, $45°$ and $60°$, use the compound angle identities to find the exact value of:

(a) $\sin 75°$ (b) $\cos 105°$ (c) $\tan(-15°)$

Working: (a)
$$\sin 75° = \sin(45° + 30°)$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

(b)
$$\cos 105° = \cos(60° + 45°)$$
$$= \cos 60° \cos 45° - \sin 60° \sin 15°$$
$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$
$$= \frac{1}{4}\left(\sqrt{2} - \sqrt{6}\right)$$

(c)
$$\tan(-15°) = \tan(30° - 45°)$$
$$= \frac{\tan 30° - \tan 45°}{1 + \tan 30° \tan 45°}$$
$$= \frac{\frac{\sqrt{3}}{3} - 1}{1 + \frac{\sqrt{3}}{3} \times 1}$$
$$= \frac{\sqrt{3} - 3}{3 + \sqrt{3}}$$
$$= \frac{\sqrt{3} - 3}{3 + \sqrt{3}} \times \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$
$$= \frac{6\sqrt{3} - 9 - 3}{9 - 3}$$
$$= \frac{6\sqrt{3} - 12}{6}$$
$$= \sqrt{3} - 2$$

E.g. 5 Express $\cos\left(x-\frac{\pi}{3}\right)$ in terms of $\cos x$ and $\sin x$.

Working:

$$\begin{aligned}\cos\left(x-\frac{\pi}{3}\right) &= \cos x\cos\frac{\pi}{3}+\sin x\sin\frac{\pi}{3}\\ &= \cos x\times\frac{1}{2}+\sin x\times\frac{\sqrt{3}}{2}\\ &= \frac{1}{2}\left(\cos x+\sqrt{3}\sin x\right)\end{aligned}$$

E.g. 6 Given that $\sin A=\frac{8}{17}$ and $\sin B=\frac{12}{13}$, where $\frac{\pi}{2}<A<\pi$ and $0<B<\frac{\pi}{2}$, find the exact value of:

(a) $\sin(A+B)$

(b) $\cos(A+B)$

Working:

(a) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

The values of $\cos A$ and $\cos B$ are needed.

$$\cos^2 A = 1-\sin^2 A = 1-\left(\frac{8}{17}\right)^2=\frac{225}{289}$$

$$\cos A=\pm\sqrt{\frac{225}{289}}=\pm\frac{15}{18}$$

Since $\frac{\pi}{2}<A<\pi$, $\cos A=-\frac{15}{17}$

$$\cos^2 B = 1-\sin^2 B = 1-\left(\frac{12}{13}\right)^2=\frac{25}{169}$$

$$\cos B=\pm\sqrt{\frac{25}{169}}=\pm\frac{5}{13}$$

Since $0<B<\frac{\pi}{2}$, $\cos B=\frac{5}{13}$

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ &= \frac{8}{17}\times\frac{5}{13}+\left(-\frac{15}{17}\right)\times\frac{12}{13}\\ &= -\frac{140}{221}\end{aligned}$$

(b)

$$\begin{aligned}\cos(A+B) &= \cos A\cos B-\sin A\sin B\\ &= \left(-\frac{15}{17}\right)\times\frac{5}{13}-\frac{8}{17}\times\frac{12}{13}\\ &= -\frac{171}{221}\end{aligned}$$

E.g. 7 Given that $\tan(x + 45°) = 2$, find the value of $\tan x$ without using a calculator.

Working: $\tan(x + 45°) = 2$:

$$\frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 2$$
$$\frac{\tan x + 1}{1 - \tan x} = 2$$
$$\tan x + 1 = 2 - 2\tan x$$
$$\tan x = \frac{1}{3}$$

E.g. 8 Solve the equation $\sin(\theta + 45°) = \sqrt{2}\cos\theta$ for $0° \le \theta \le 360°$.

Working:

$$\sin(\theta + 45°) = \sqrt{2}\cos\theta$$
$$\sin\theta\cos 45° + \cos\theta\sin 45° = \sqrt{2}\cos\theta$$

Multiply by $\frac{2}{\sqrt{2}}$: $\sin\theta \times \frac{\sqrt{2}}{2} + \cos\theta \times \frac{\sqrt{2}}{2} = \sqrt{2}\cos\theta$

$$\sin\theta + \cos\theta = 2\cos\theta$$
$$\sin\theta = \cos\theta$$

Since $\frac{\sin\theta}{\cos\theta} = \tan\theta$: $\tan\theta = 1$

$\theta = 45°$ or $\theta = 180° + 45°$

$\theta = 45°$ or $\theta = 225°$

Video: Compound angle identities

Video: Compound angle identities (exact values)

Video: Compound angle identities (proving identities)

Video: Compound angle identities (equations)

Solutions to Starter and E.g.s

Exercise

p167 8A Qu 1ac, 2-8, (9 red)