Definite Integration by Substitution

Starter

1. **(Review of last lesson)** Use a suitable substitution to integrate $\int \frac{2x+1}{(x-3)^6}dx$.

Working: Let $u = x - 3 \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad dx = du$

$$\int \frac{2x+1}{(x-3)^6}dx = \int \frac{2x+1}{u^6}du \qquad \textit{replace } x-3 \textit{ and } dx$$

$$= \int \frac{2(u+3)+1}{u^6}du \qquad \textit{since } x = u+3$$

$$= \int \frac{2u+7}{u^6}du$$

$$= \int (2u^{-5} + 7u^{-6})du \qquad \textit{form 2 fractions}$$

$$= \frac{2u^{-4}}{(-4)} + \frac{7u^{-5}}{(-5)} + c \qquad \textit{integrate with respect to } u$$

$$= -\frac{1}{2u^4} - \frac{7}{5u^5} + c$$

$$= -\frac{1}{10u^5}\left(5u + 14\right) + c$$

$$= -\frac{1}{10(x-3)^5}\left(5(x-3) + 14\right) + c \qquad \textit{replace } u$$

$$= -\frac{1}{10(x-3)^5}\left(5x - 1\right) + c$$

$$= \frac{1-5x}{10(x-3)^5} + c$$

E.g. 1 Use the substitution $u = x + 3$ to find $\int_{-2}^{1} 2x(x+3)^4 dx$.

Working: Let $u = x + 3$ so $x = u - 3$

$\frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$

Table of changed values:

When $x = 1, u = 1 + 3 = 4$

When $x = -2, u = -2 + 3 = 1$

x	u
1	4
−2	1

$$\int_{-2}^{1} 2x(x+3)^4 dx = \int_{1}^{4} 2(u-3)u^4 du$$ ***replace $x+3$, dx and limits***

$$= \int_{1}^{4} 2u^5 - 6u^4 du$$ ***expand brackets***

$$= \left[\frac{u^6}{3} - \frac{6u^5}{5}\right]_1^4$$ ***integrate with respect to u***

$$= \left(\frac{4^6}{3} - \frac{6 \times 4^5}{5}\right) - \left(\frac{1}{3} - \frac{6}{5}\right)$$ ***substitute***

$$= \frac{687}{5} = 137.4$$

E.g. 2 Find: (a) $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}dx$ (b) $\displaystyle\int_1^5 \frac{x}{\sqrt{3x+1}}dx$

Working: (a) Let $u = 1 + e^x$ $\Rightarrow$ $\dfrac{du}{dx} = e^x$ $\Rightarrow$ $\dfrac{du}{e^x} = dx$

Table of changed values:
When $x = 1, u = 1 + e^1 = 1 + e$
When $x = 0, u = 1 + e^0 = 2$ ***since $e^0 = 1$***

x	u
1	1 + e
0	2

$$\int_0^1 \frac{e^x}{(1+e^x)^2}dx = \int_2^{1+e} \frac{e^x}{u^2}\frac{du}{e^x}$$ ***replace $1+e^x$, dx and limits***

$$= \int_2^{1+e} u^{-2}du$$ ***cancel e^x and transform***

$$= \left[\frac{u^{-1}}{(-1)}\right]_2^{1+e}$$ ***integrate with respect to u***

$$= \left[-\frac{1}{u}\right]_2^{1+e}$$ *** ***simplify***

$$= \left(-\frac{1}{1+e}\right) - \left(-\frac{1}{2}\right)$$ ***substitute***

$$= \frac{1}{2} - \frac{1}{1+e} = \frac{e-1}{2(1+e)}$$

N.B. *** $= \left[\dfrac{1}{u}\right]_{1+e}^{2}$ also possible. Remember: changing the limits around changes the sign of the integral.

(b) Let $u = 3x + 1 \quad \Rightarrow \quad \frac{du}{dx} = 3 \quad \Rightarrow \quad \frac{du}{3} = dx$

Table of changed values:
When $x = 5$, $u = 3 \times 5 + 1 = 16$
When $x = 1$, $u = 3 \times 1 + 1 = 4$

x	u
5	16
1	4

$$\int_1^5 \frac{x}{\sqrt{3x+1}}dx = \int_4^{16} \frac{x}{3\sqrt{u}}du \quad \textit{replace } 3x+1, dx \textit{ and limits}$$

$$= \int_4^{16} \frac{u-1}{9\sqrt{u}}du \quad \textit{since } x = \frac{u-1}{3}$$

$$= \frac{1}{9}\int_4^{16} \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \quad \textit{form 2 fractions}$$

$$= \frac{1}{9}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right]_4^{16} \quad \textit{integrate with respect to } u$$

$$= \frac{1}{9}\left(\frac{2 \times 16^{\frac{3}{2}}}{3} - 2 \times 16^{\frac{1}{2}}\right) - \frac{1}{9}\left(\frac{2 \times 4^{\frac{3}{2}}}{3} - 2 \times 4^{\frac{1}{2}}\right)$$

$$= \frac{100}{27} = 3\frac{19}{27} = 3.703703...$$

Video: Integration by substitution
Video: Integration by substitution involving square roots

Integration by substitution EQ

Solutions to Starter and E.g.s

Exercise
p229 11C Qu 4i, 5-8