Describing Motion in 2-D

Starter

1. **(Review of last lesson)** In a chemical reaction, a chemical X has a concentration of x moles per litre. x is known to satisfy the differential equation $\frac{dx}{dt} = k(a-x)^2$, where a and k are constants. Given that the initial concentration is zero, find a formula for x at time t, expressing your answer in terms of a and k.

Working:

$$\frac{dx}{dt} = k(a-x)^2 \quad \Rightarrow \quad \int \frac{1}{(a-x)^2}dx = \int k\,dt$$

$$\int (a-x)^{-2}dx = \int k\,dt$$

$$(a-x)^{-1} = kt + c$$

$$\frac{1}{a-x} = kt + c$$

When $t = 0, x = 0$: $\quad \frac{1}{a} = c$

Substituting:

$$\frac{1}{a-x} = kt + \frac{1}{a}$$

$$\frac{1}{a-x} = \frac{akt+1}{a}$$

$$a-x = \frac{a}{akt+1}$$

$$x = a - \frac{a}{akt+1}$$

$$x = \frac{a(akt+1)-a}{akt+1}$$

$$x = \frac{a^2kt}{akt+1}$$

E.g. 1 A particle travels from the point $A\begin{pmatrix}-5\\-3\end{pmatrix}$ to $B\begin{pmatrix}7\\2\end{pmatrix}$ in 2 seconds.

It then travels from $B\begin{pmatrix}7\\2\end{pmatrix}$ to $C\begin{pmatrix}4\\6\end{pmatrix}$ in 4 seconds. Calculate:

(a) the displacement and the average velocity from A to B
(b) the distance travelled and the average speed from B to C
(c) the average velocity from A to C
(d) the average speed from A to C.

Working:

(a) The vector from A to B is $\mathbf{b}-\mathbf{a}$

So displacement from A to B is $\begin{pmatrix}7\\2\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix}=\begin{pmatrix}12\\5\end{pmatrix}$

$$\text{Average velocity} = \frac{\begin{pmatrix}12\\5\end{pmatrix}}{2} = \begin{pmatrix}6\\2.5\end{pmatrix}$$

(b) Distance between two points B and C is $\left|\overrightarrow{BC}\right| = \left|\mathbf{c}-\mathbf{b}\right|$

$$\text{Distance} = \left|\begin{pmatrix}4\\6\end{pmatrix}-\begin{pmatrix}7\\2\end{pmatrix}\right| = \left|\begin{pmatrix}-3\\4\end{pmatrix}\right| = \sqrt{(-3)^2+4^2} = 5$$

$$\text{Average speed} = \frac{5}{4} = 1.25 \text{ m/s}$$

(c) Displacement from A to C is $\begin{pmatrix}4\\6\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix}=\begin{pmatrix}9\\9\end{pmatrix}$

$$\text{Average velocity} = \frac{\begin{pmatrix}9\\9\end{pmatrix}}{2+4} = \begin{pmatrix}1.5\\1.5\end{pmatrix}$$

(d) $$\text{Distance from } A \text{ to } B = \left|\begin{pmatrix}12\\5\end{pmatrix}\right| = \sqrt{12^2+5^2} = 13$$

Distance from A to $B = 5$

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{time}}$$
$$= \frac{13+5}{2+4}$$
$$= 3 \text{ m/s}$$

Further examples
Read worked example 19.2 on p427
Read worked example 19.4 on p428

Solutions to Starter and E.g.s

Exercise
p429 19A Qu 1i, 2i, 3-8