Differentiating Parametric Equations

Starter

1. **(Review of last lesson)**
Transform the parametric curve $x = \tan\theta$, $y = \sec\theta$ into cartesian form.

Working: From the identity $1 + \tan^2 x = \sec^2 x$ we get $y^2 = 1 + x^2$

2. Given that $x = 5t$, $y = t^2$, express $\frac{dy}{dx}$ in terms of t.

Hint: remember the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Working: $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{5} = \frac{2t}{5}$$

E.g. 1 Find $\frac{dy}{dx}$ in terms of t for the following curves:

(a) $x = e^{3t}$, $y = 4t^3 - 2t^2$ (b) $x = \cos t - \sin 5t$, $y = 4\sin t$

Working: (a) $\frac{dx}{dt} = 3e^{3t}$ and $\frac{dy}{dt} = 12t^2 - 4t = 4t(3t - 1)$

Using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$: $\frac{dy}{dx} = \frac{4t(3t-1)}{3e^{3t}}$

(b) $\frac{dx}{dt} = -\sin t - 5\cos 5t$ and $\frac{dy}{dt} = 4\cos t$

Using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$: $\frac{dy}{dx} = \frac{4\cos t}{-\sin t - 5\cos 5t}$

$$\frac{dy}{dx} = -\frac{4\cos t}{\sin t + 5\cos 5t}$$

E.g. 2 The curve C is defined by the parametric equations $x = t^2 - 1$ and $y = t^3 - 3t + 4$.

(a) Find $\dfrac{dy}{dx}$ in terms of t.

(b) Find the coordinates of the turning points.

Working: (a) $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 3t^2 - 3 = 3(t^2 - 1)$

Using $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$: $\qquad \dfrac{dy}{dx} = \dfrac{3(t^2 - 1)}{2t}$

(b) A SP occurs when $\dfrac{dy}{dx} = 0$ so $\dfrac{3(t^2 - 1)}{2t} = 0$

If $\dfrac{f(x)}{g(x)} = 0$ then $f(x) = 0$: $\qquad 3(t^2 - 1) = 0$

$t^2 - 1 = 0 \quad \Rightarrow \quad t = \pm 1$

When $t = 1$: $\quad x = 1^2 - 1 = 0$

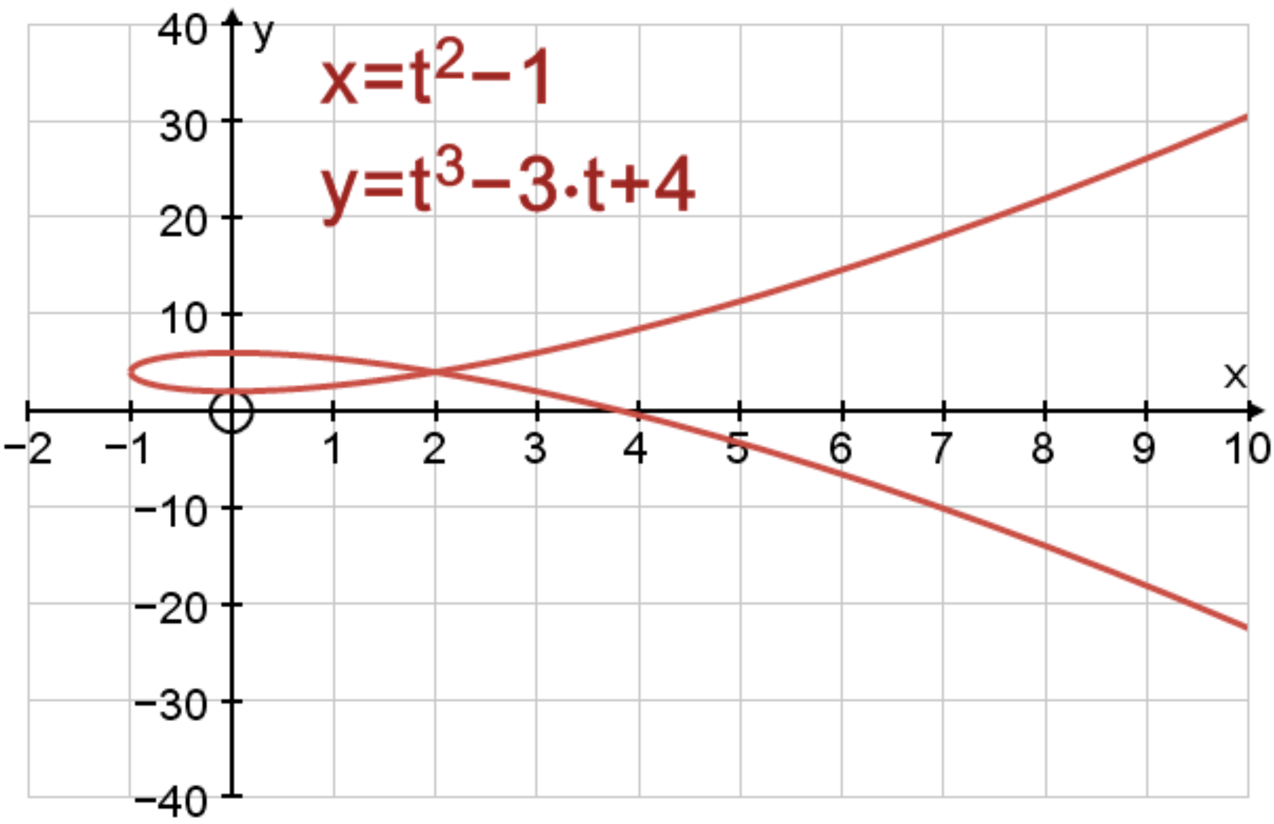
$y = 1^3 - 3 \times 1 + 4 = 2$

When $t = 1$: $\quad x = (-1)^2 - 1 = 0$

$y = (-1)^3 - 3 \times (-1) + 4 = 6$

Turning points are at $(0, 2)$ and $(0, 6)$.

The graph is below.



E.g. 3 Find the equation of the normal of the curve $x = \sin 2t$, $y = t\cos t + 2\sin t$ at $t = \pi$.

Working: $\frac{dx}{dt} = 2\cos 2t$ and $\frac{dy}{dt} = \cos t - t\sin t + 2\cos t = 3\cos t - t\sin t$

Using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$: $\quad \frac{dy}{dx} = \frac{3\cos t - t\sin t}{2\cos 2t}$

When $t = \pi$, $\frac{dy}{dx} = \frac{3\cos\pi - \pi\sin\pi}{2\cos 2\pi} = \frac{-3-0}{2} = -\frac{3}{2}$ ≡ ***tangent***

Gradient of normal is $\frac{2}{3}$

When $t = \pi$: $\quad x = \sin 2\pi = 0$

$y = \pi\cos\pi + 2\sin\pi = -\pi$

Using $y - y_1 = m(x - x_1)$: $\quad y - (-\pi) = \frac{2}{3}(x - 0)$

The equation of the normal is $y = \frac{2}{3}x - \pi$.

Video: Differentiating parametric functions
Video: Parametric functions (tangents and normals)

Solutions to Starter and E.g.s

Exercise
p262 12C Qu 1i, 2i, 3-8