Differentiation of a^x

Starter

1. **(Review of last lesson)**
Find the gradient of the curve $e^x + 2\ln y = y^3$ at the point where $y = 1$.

Working: When $y = 1$, $e^x + 2\ln 1 = 1 \quad \Rightarrow \quad e^x = 1$ since $\ln 1 = 0$
So $x = 0$

Differentiate w.r.t. x: $e^x + \frac{2}{y}\frac{dy}{dx} = 3y^2\frac{dy}{dx}$

Substitute (0, 1): $e^0 + 2\frac{dy}{dx} = 3\frac{dy}{dx}$

$\frac{dy}{dx} = 1$

N.B. w.r.t. is short for "with respect to"

2. **(Review of last lesson)** If $\tan y = 2x$, find $\frac{dy}{dx}$ in terms of x.

Working: Differentiate w.r.t. x: $\sec^2 y\frac{dy}{dx} = 2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sec^2 y}$

But $1 + \tan^2 y = \sec^2 y$ so $\frac{dy}{dx} = \frac{2}{1 + \tan^2 y}$

Since $\tan y = 2x$, $\frac{dy}{dx} = \frac{2}{1 + (2x)^2} = \frac{2}{1 + 4x^2}$

E.g. 1 By firstly taking natural logarithms of both sides and then using one of the laws of logarithms, differentiate $y = 3^x$.

Working:
$\ln y = \ln 3^x$ ***taking*** $\ln$ ***of both sides***
$\ln y = x\ln 3$ ***using the 3rd law of logs***
N.B. $\ln 3$ is now the coefficient of x
$\frac{1}{y}\frac{dy}{dx} = \ln 3$ ***differentiating with respect to*** x
$\frac{dy}{dx} = y\ln 3$ ***rearranging***
$\frac{dy}{dx} = 3^x\ln 3$ ***replace*** y ***by*** 3^x

E.g. 2 Differentiate the following functions with respect to x:
(a) $y = 7^x$ (b) $y = 5 \times 8^x$ (c) $y = 2^{x^3}$ (d) $y = x^x$

Working: (a)
$\ln y = \ln 7^x$ ***taking*** $\ln$ ***of both sides***
$\ln y = x\ln 7$ ***using the 3rd law of logs***
$\frac{1}{y}\frac{dy}{dx} = \ln 7$ ***differentiating with respect to*** x
$\frac{dy}{dx} = y\ln 7$ ***rearranging***

$$\frac{dy}{dx} = 7^x \ln 7 \qquad \textit{replace } y \textit{ by } 7^x$$

(b) $y = 5 \times 8^x$

$$\frac{y}{5} = 8^x \qquad \textit{rearrange}$$

$$\ln\left(\frac{y}{5}\right) = x \ln 8 \qquad \textit{taking } \ln \textit{ of both sides and 3rd law}$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = \ln 8 \qquad \textit{differentiate}$$

$$\frac{dy}{dx} = y \ln 8 \qquad \textit{rearrange}$$

$$\frac{dy}{dx} = 8^x 5 \ln 8 \qquad \textit{replace } y \textit{ by } 5 \times 8^x$$

(c) $y = 2^{x^3}$

$$\ln y = x^3 \ln 2 \qquad \textit{taking } \ln \textit{ of both sides and 3rd law}$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = 3x^2 \ln 2 \qquad \textit{differentiate}$$

$$\frac{dy}{dx} = 3x^2 y \ln 2 \qquad \textit{rearrange}$$

$$\frac{dy}{dx} = 3x^2 2^{x^3} \ln 2 \qquad \textit{replace } y \textit{ by } 2^{x^3}$$

(d) $y = x^x$

$$\ln y = x \ln x \qquad \textit{taking } \ln \textit{ of both sides and 3rd law}$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = 1 + \ln x \qquad \textit{differentiate using the product rule}$$

$$\frac{dy}{dx} = y(1 + \ln x) \qquad \textit{rearrange}$$

$$\frac{dy}{dx} = x^x(1 + \ln x) \qquad \textit{replace } y \textit{ by } x^x$$

In general: if $y = a^{f(x)}$ then $\frac{dy}{dx} = f'(x)a^{f(x)} \ln a$

Video (differentiating a^x): Derivatives of a^x
Video (differentiating x^x): Derivatives of x^x

Differentiation EQ

Solutions to Starter and E.g.s

Exercise
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