Distribution of the sample mean

Starter

1. $10\,\%$ of the chocolates produced in a factory are mis-shapes. A sample of 1000 chocolates is taken.
   (a) Find the probability that the number of mis-shapes is between 90 and 115 inclusive:
      (i) using a binomial distribution
      (ii) using a Normal approximation.
   (b) Calculate the percentage error of the Normal approximation.

Working: (a) $n = 1000, p = 0.1 \quad \Rightarrow \quad X \sim \text{B}(1000, 0.1)$

$$P(90 \le X \le 115) = P(X \le 115) - P(X \le 89)$$
$$= 0.9465 - 0.1334$$
$$= 0.8131$$

(b) $np = 100, np(1-p) = 90$

$X \sim \text{B}(1000, 0.1) \quad \approx \quad X \sim \text{N}(100, 90)$

$P(90 \le X \le 115) = 0.7972$

(c) Percentage error $= \dfrac{0.8131 - 0.7972}{0.8131} \times 100\,\% = 1.96\,\%$ (3 s.f.)

E.g. 1 (a) If $X \sim \text{N}(75, 8)$, find $P(\bar{X}_9 < 74)$ (b) If $X \sim \text{N}(12, 3^2)$, find $P(\bar{X}_7 > 12.5)$.

Working: (a) $X \sim \text{N}(75, 8) \quad \Rightarrow \quad \bar{X}_9 \sim \text{N}\left(75, \dfrac{8}{9}\right)$

Standard error $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{2\sqrt{2}}{3}$

$P(\bar{X}_9 < 74) = 0.144$ (3 s.f.)

(b) $X \sim \text{N}(12, 3^2) \quad \Rightarrow \quad \bar{X}_7 \sim \text{N}\left(12, \dfrac{3^2}{7}\right)$

Standard error $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{3}{\sqrt{7}} = \dfrac{3\sqrt{7}}{7}$

$P(\bar{X}_7 > 12.5) = 1 - P(\bar{X}_7 < 12.5) = 0.330$ (3 s.f.)

E.g. 2 A random sample of size 15 is taken from a normal distribution with mean 60 and standard deviation 4. Find the probability that the mean of the sample is less than 58.

Working: $X \sim \text{N}(60, 4^2) \quad \Rightarrow \quad \bar{X}_{15} \sim \text{N}\left(60, \dfrac{4^2}{15}\right)$

Standard error $= \dfrac{\sigma}{\sqrt{n}} = \dfrac{4}{\sqrt{15}}$

$P(\bar{X} < 58) = 0.0264$

E.g. 3 The heights of a particular species of plant follow a normal distribution with mean 21 and standard deviation $\sqrt{90}$. A random sample of 10 plants is taken and the mean height calculated. Find the probability that this mean lies between 18 cm and 27 cm.

Working: $X \sim \text{N}(21, 90) \Rightarrow \bar{X}_{10} \sim \text{N}\left(21, \frac{90}{10}\right) = \text{N}(21, 9)$

Standard error $= \frac{\sigma}{\sqrt{n}} = 3$

$P(18 < \bar{X} < 27) = 0.8186$

E.g. 4 A large number of random samples of size n are taken from the distribution of X where $X \sim \text{N}(74, 36)$ and the sample means are calculated. If $P(\bar{X} > 72) = 0.854$, estimate the value of n.

Working: $X \sim \text{N}(74, 36) \Rightarrow \bar{X}_n \sim \text{N}\left(74, \frac{36}{n}\right)$

Standard error $= \frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{n}}$

$P(\bar{X} > 72) = 0.854 \quad \Rightarrow \quad P(\bar{X} < 72) = 1 - 0.854 = 0.146$

Let z be the $z-$value corresponding to 72.

$P(Z < z) = 0.146 \quad \Rightarrow \quad z = -1.0537$

Substitute into $z = \frac{x - \mu}{\frac{\sigma}{\sqrt{n}}}$: $\quad -1.0537 = \frac{72 - 74}{\frac{6}{\sqrt{n}}}$

$-1.0537 \times 6 = -2 \times \sqrt{n}$

$n = 9.99 \approx 10$

The value of n is 10.

E.g. 5 A normal distribution has a mean of 30 and a variance of 5. Find n such that the probability that the average of n observations exceeds 30.5 is less than $1\ \%$.

Working: $X \sim \text{N}(30, 5) \Rightarrow \bar{X}_n \sim \text{N}\left(30, \frac{5}{n}\right)$

Standard error $= \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{5}{n}}$

$P(\bar{X} > 30.5) < 0.01 \quad \Rightarrow \quad P(\bar{X} < 30.5) > 1 - 0.01 = 0.99$

Let z be the $z-$value corresponding to 30.5.

$P(Z < z) = 0.99 \quad \Rightarrow \quad z = 2.326$

Substitute into $z = \frac{x - \mu}{\frac{\sigma}{\sqrt{n}}}$: $\quad 2.326 = \frac{30.5 - 30}{\sqrt{\frac{5}{n}}}$

$2.326 \times \sqrt{5} = 0.5 \times \sqrt{n}$

$n \approx 108.2$

The value of n is greater than 108.

Video: **Distribution of sample mean A**
Video: **Distribution of sample mean B**

Solutions to Starter and E.g.s

Exercise
p400 18A Qu 1i, 2i, 3-8,(9-10 red)