Double angle identities

Starter

1. Given that $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{12}{13}$, where A is obtuse and B is acute, find the exact value of:
(a) $\sin(A - B)$ (b) $\cos(A + B)$

Working: (a)

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

The values of $\cos A$ and $\sin B$ are needed.

$\cos^2 A = 1 - \sin^2 A = 1 - \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25}$

$\cos A = \pm\sqrt{\dfrac{16}{25}} = \pm\dfrac{4}{5}$

Since A is obtuse, $\cos A = -\dfrac{4}{5}$

$\sin^2 B = 1 - \cos^2 B = 1 - \left(\dfrac{12}{13}\right)^2 = \dfrac{25}{169}$

$\sin B = \pm\sqrt{\dfrac{25}{169}} = \pm\dfrac{5}{13}$

Since B is acute, $\sin B = \dfrac{5}{13}$

$$\begin{aligned}\sin(A - B) &= \sin A \cos B - \cos A \sin B\\ &= \frac{3}{5} \times \frac{12}{13} - \left(-\frac{4}{5}\right) \times \frac{5}{13}\\ &= \frac{56}{65}\end{aligned}$$

(b)

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B\\ &= \left(-\frac{4}{5}\right) \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}\\ &= -\frac{63}{65}\end{aligned}$$

2. Using the compound angle identities and by setting B equal to A, find expressions for:
(a) $\sin 2A$ (b) $\cos 2A$ (c) $\tan 2A$

Working: (a)

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A + A) = \sin A \cos A + \cos A \sin A$

$\therefore \sin 2A = 2 \sin A \cos A$

(b)

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A + A) = \cos A \cos A - \sin A \sin A$

$\therefore \cos 2A = \cos^2 A - \sin^2 A$

(c)

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

$$\therefore \tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

E.g. 1 Starting with $\cos 2A \equiv \cos^2 A - \sin^2 A$ and using the identity involving $\cos A$ and $\sin A$, find an identity for $\cos 2A$ involving:

(a) $\cos A$ (b) $\sin A$

Working: (a) $\cos^2 A + \sin^2 A \equiv 1 \quad \Rightarrow \quad \sin^2 A \equiv 1 - \cos^2 A$

Substituting: $\cos 2A \equiv \cos^2 A - (1 - \cos^2 A)$

$\cos 2A \equiv 2\cos^2 A - 1$

(b) $\cos^2 A + \sin^2 A \equiv 1 \quad \Rightarrow \quad \cos^2 A \equiv 1 - \sin^2 A$

Substituting: $\cos 2A \equiv (1 - \sin^2 A) - \sin^2 A$

$\cos 2A \equiv 1 - 2\sin^2 A$

E.g. 2 Solve these equations for $0° \le \theta \le 360°$:

(a) $\sin 2\theta = \cos\theta$ (b) $2\cos 2\theta = 5 - 13\sin\theta$

(c) $\tan 2\theta + \tan\theta = 0$ (d) $\sin 2\theta = 2\cos 2\theta$

If not exact, give your angles to 3 s.f..

Working: (a) Using $\sin 2\theta = 2\cos\theta\sin\theta$: $2\cos\theta\sin\theta = \cos\theta$

$2\cos\theta\sin\theta - \cos\theta = 0$

N.B. ***Do not divide by*** $\cos\theta$ otherwise solutions will be lost.

Factorise: $\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0 \quad \Rightarrow \quad \theta = 90°$ or $270°$

$2\sin\theta - 1 = 0 \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 30°$ or $150°$

So $\theta = 30°, 90°, 150°$ or $270°$.

(b) ***We could replace*** $\cos 2\theta$ ***by either*** $\cos 2\theta \equiv 2\cos^2\theta - 1$ ***or*** $\cos 2\theta \equiv 1 - 2\sin^2\theta$.

Since there is a term in $\sin\theta$ ***in the equation, we choose*** $\cos 2\theta \equiv 1 - 2\sin^2\theta$ ***so that we can form a quadratic in*** $\sin\theta$.

$2\cos 2\theta = 5 - 13\sin\theta \quad \Rightarrow \quad 2(1 - 2\sin^2\theta) = 5 - 13\sin\theta$

Expand and collect like terms: $4\sin^2\theta - 13\sin\theta + 3 = 0$

Factorising: $(4\sin\theta - 1)(\sin\theta - 3) = 0$

$\sin\theta - 3 = 0 \Rightarrow$ No solution

or $4\sin\theta - 1 = 0 \quad \Rightarrow \quad \sin\theta = \frac{1}{4}$

$\Rightarrow \quad \theta = 14.5°$ or $180° - 14.5° = 166°$ (3 s.f.)

(c) $\tan 2\theta + \tan\theta = 0 \quad \Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta = 0$

Multiply by $1 - \tan^2\theta$: $2\tan\theta + \tan\theta(1 - \tan^2\theta) = 0$

Expand and factorise: $\tan\theta(3 - \tan^2\theta) = 0$

$\tan\theta = 0 \quad \Rightarrow \quad \theta = 0°, 180°$ and $360°$

$3 - \tan^2\theta = 0 \quad \Rightarrow \quad \tan\theta = \pm\sqrt{3}$

$\Rightarrow \quad \theta = 60°, 120°, 240°$ or $300°$

So $\theta = 0°, 60°, 120°, 180°, 240°, 300°$ or $360°$.

(d) $\sin 2\theta = 2\cos 2\theta \quad \Rightarrow \quad \dfrac{\sin 2\theta}{\cos 2\theta} = 2$

N.B. Double angle identities are not necessary.

$\tan 2\theta = 2$

$2\theta = 63.4^\circ, 63.4^\circ + 180^\circ, 63.4^\circ + 360^\circ$ or $63.4^\circ + 180^\circ + 360^\circ$

$\theta = 31.7^\circ, 122^\circ, 212^\circ$ or 302° (3 s.f.)

When proving identities, it is usual to start with the expression on the left-hand side and to manipulate it over a series of steps until it becomes the expression on the right-hand side.

E.g. 3 Prove the identities:

(a) $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$ (b) $\cos 3A \equiv 4\cos^3 A - 3\cos A$

Working: (a) $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \dfrac{\cos^2 A - \sin^2 A}{\cos A + \sin A}$

$\equiv \dfrac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A + \sin A}$

$\equiv \cos A - \sin A$

(b) $\cos 3A \equiv \cos(2A + A)$

$\equiv \cos 2A \cos A - \sin 2A \sin A$

$\equiv (2\cos^2 A - 1)\cos A - 2\sin A \cos A \sin A$

N.B. The final answer is in terms of $\cos A$ so we use the identity $\cos 2A \equiv 2\cos^2 A - 1$.

$\equiv 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$\equiv 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$\equiv 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$\equiv 4\cos^3 A - 3\cos A$

E.g. 4 Find the values of $\sin 2\theta$ and $\cos 2\theta$ given that $\sin\theta = \dfrac{3}{5}$ and θ is obtuse.

Working: $\sin\theta = \dfrac{3}{5} \quad \Rightarrow \quad \cos^2\theta = 1 - \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25}$

Since θ is obtuse, $\cos\theta = -\dfrac{4}{5}$

$\sin 2\theta \equiv 2\sin\theta\cos\theta = 2 \times \dfrac{3}{5} \times \left(-\dfrac{4}{5}\right) = -\dfrac{24}{25}$

$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta = \left(-\dfrac{4}{5}\right)^2 - \left(\dfrac{3}{5}\right)^2 = \dfrac{7}{25}$

Video: Double angle identities (proving identities)
Video: Double angle identities (solving equations)

Double angle identities EQ

Solutions to Starter and E.g.s

Exercise
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