

Equation of the Trajectory

Starter

1. **(Review of last lesson)** At the point of being hit, a golf ball has an initial speed of 22 m/s. It lands 4 s later. Find its angle of projection.

Hint: how long does the ball take to reach its maximum height?

Working: Let θ be the angle of projection.
 The time to reach the maximum height is 2 s
 $u_y = 22 \sin \theta, v_y = 0, a_y = -9.8, t = 2$
 No $s \Rightarrow v = u + at: 0 = 22 \sin \theta - 9.8 \times 2$
 $\sin \theta = \frac{9.8 \times 2}{22} = 63.0^\circ$
 The angle of projection is 63.0°

2. Show that a particle projected from the ground with velocity u at an angle of θ to the horizontal:

- (a) reaches its maximum height when $t = \frac{u \sin \theta}{g}$.
- (b) reaches a maximum height of $\frac{u^2 \sin^2 \theta}{2g}$
- (c) hits the ground when $t = \frac{2u \sin \theta}{g}$.
- (d) has a range of $\frac{u^2 \sin 2\theta}{g}$
- (e) Given that the range, R , is given by $R = \frac{u^2 \sin 2\theta}{g}$, prove by differentiation that the greatest range is when $\theta = 45^\circ$. Remember u and g are constants.

Working:

(a) Maximum height $\Rightarrow v_y = 0$
 $u_y = u \sin \theta, v_y = 0, a_y = -g, t = ?$
 No $s \Rightarrow v = u + at: 0 = u \sin \theta - gt$
 Rearranging gives $t = \frac{u \sin \theta}{g}$

(b) $u_y = u \sin \theta, v_y = 0, a_y = -g, s_y = ?$
 No $t \Rightarrow v^2 = u^2 + 2as: 0 = (u \sin \theta)^2 + 2 \times (-g) \times s_y$
 Rearranging gives $\frac{u^2 \sin^2 \theta}{2g}$

(c) Reaches the ground $\Rightarrow s_y = 0$
 $u_y = u \sin \theta, s_y = 0, a_y = -g, t = ?$
 No $v \Rightarrow s = ut + \frac{1}{2}at^2: ut \sin \theta + \frac{1}{2} \times (-g) \times t^2 = 0$
 Multiply by 2 and factorise: $t(2u \sin \theta - gt) = 0$
 Either $t = 0$ (at projection) or $2u \sin \theta - gt = 0$ (hits ground)
 Rearranging gives $t = \frac{2u \sin \theta}{g}$

(d) Range $\Rightarrow s_x$
 $u_x = u \cos \theta, a_x = 0, t = \frac{2u \sin \theta}{g}, s_x = ?$
 No $v \Rightarrow s = ut + \frac{1}{2}at^2: s_x = u \cos \theta \times \frac{2u \sin \theta}{g} + 0$
 $s_x = \frac{2u^2 \sin \theta \cos \theta}{g}$
 Since $\sin 2\theta = 2 \sin \theta \cos \theta$, the range is $\frac{u^2 \sin 2\theta}{g}$.

(e) $R = \frac{u^2 \sin 2\theta}{g} \Rightarrow \frac{dR}{d\theta} = \frac{2u^2 \cos 2\theta}{g}$
 A maximum occurs when $\frac{dR}{d\theta} = 0: \frac{2u^2 \cos 2\theta}{g} = 0$
 i.e. $\cos 2\theta = 0 \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$
 The greatest range is when $\theta = 45^\circ$.

E.g. 1 A projectile is fired from the origin with speed 3 m/s, at an angle α above the horizontal. The projectile passes through the point (3, 1) m. Show that $1 = 3 \tan \alpha - \frac{g}{2 \cos^2 \alpha}$.

Hint: use the equation for the trajectory.

Working: Substituting into $y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$:
 $1 = 3 \tan \alpha - \frac{g \times 3^2}{2 \times 3^2 \times \cos^2 \alpha}$
 Simplifying gives $1 = 3 \tan \alpha - \frac{g}{2 \cos^2 \alpha}$

E.g. 2 A golf ball is struck with an angle of 60° and passes through the point (4, 2). What was the initial speed of the ball?

Working: Substituting into $y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$:
 $2 = 4 \tan 60 - \frac{g \times 4^2}{2u^2 \cos^2 60}$
 $2 = 4\sqrt{3} - \frac{8g}{\frac{1}{4}u^2}$ since $\tan 60 = \sqrt{3}$ and $\cos 60 = \frac{1}{2}$
 $\frac{32g}{u^2} = 4\sqrt{3} - 2 \Rightarrow u^2 = \frac{32g}{4\sqrt{3} - 2} \Rightarrow u = 7.98$
 The initial speed of the ball 7.98 m/s (3 s.f.)

Video: [Equation of trajectory](#)
Video: [Finding speed and direction](#)

Exercise

p460 20B Qu 1-3, 4-6 (red)

