Equation of the Trajectory

Starter

1. (Review of last lesson) At the point of being hit, a golf ball has an initial speed of 22 m/s. It lands 4 s later. Find its angle of projection.
Hint: how long does the ball take to reach its maximum height?

Working: Let θ be the angle of projection.
The time to reach the maximum height is 4 s
$u_y = 22\sin\theta,\ v_y = 0,\ a_y = -9.8,\ t = 2$
No s $\Rightarrow$ $v = u + at$: $0 = 22\sin\theta - 9.8\times 2$
$\sin\theta = \dfrac{9.8\times 2}{22} = 63.0^{\circ}$
The angle of projection is 63.0°

2. Show that a particle projected from the ground with velocity u at an angle of θ to the horizontal:

(a) reaches its maximum height when $t = \dfrac{u\sin\theta}{g}$.

(b) reaches a maximum height of $\dfrac{u^2\sin^2\theta}{2g}$

(c) hits the ground when $t = \dfrac{2u\sin\theta}{g}$.

(d) has a range of $\dfrac{u^2\sin 2\theta}{g}$

(e) Given that the range, R, is given by $R = \dfrac{u^2\sin 2\theta}{g}$, prove by differentiation that the greatest range is when $\theta = 45^{\circ}$. Remember u and g are constants.

Working:

(a) Maximum height $\Rightarrow$ $v_y = 0$
$u_y = u\sin\theta,\ v_y = 0,\ a_y = -g,\ t = ?$
No s $\Rightarrow$ $v = u + at$: $0 = u\sin\theta - gt$
Rearranging gives $t = \dfrac{u\sin\theta}{g}$

(b) $u_y = u\sin\theta,\ v_y = 0,\ a_y = -g,\ s_y = ?$
No t $\Rightarrow$ $v^2 = u^2 + 2as$: $0 = (u\sin\theta)^2 + 2\times(-g)\times s_y$
Rearranging gives $\dfrac{u^2\sin^2\theta}{2g}$

(c) Reaches the ground $\Rightarrow$ $s_y = 0$
$u_y = u\sin\theta,\ s_y = 0,\ a_y = -g,\ t = ?$
No v $\Rightarrow$ $s = ut + \dfrac{1}{2}at^2$: $ut\sin\theta + \dfrac{1}{2}\times(-g)\times t^2 = 0$
Multiply by 2 and factorise: $t\left(2u\sin\theta - gt\right) = 0$
Either $t = 0$ (at projection) or $2u\sin\theta - gt = 0$ (hits ground)
Rearranging gives $t = \dfrac{2u\sin\theta}{g}$

(d) Range $\Rightarrow$ s_x

$u_x = u\cos\theta, a_x = 0, t = \dfrac{2u\sin\theta}{g}, s_x = ?$

No v $\Rightarrow$ $s = ut + \dfrac{1}{2}at^2$: $s_x = u\cos\theta \times \dfrac{2u\sin\theta}{g} + 0$

$$s_x = \frac{2u^2\sin\theta\cos\theta}{g}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, the range is $\dfrac{u^2\sin 2\theta}{g}$.

(e) $R = \dfrac{u^2\sin 2\theta}{g}$ $\Rightarrow$ $\dfrac{dR}{d\theta} = \dfrac{2u^2\cos 2\theta}{g}$

A maximum occurs when $\dfrac{dR}{d\theta} = 0$: $\dfrac{2u^2\cos 2\theta}{g} = 0$

i.e. $\cos 2\theta = 0$ $\Rightarrow$ $2\theta = 90^o$ $\Rightarrow$ $\theta = 45^o$

The greatest range is when $\theta = 45^o$.

E.g. 1 A projectile is fired from the origin with speed 3 m/s, at an angle α above the horizontal. The projectile passes through the point $(3, 1)$ m. Show that $1 = 3\tan\alpha - \dfrac{g}{2\cos^2\alpha}$.

Hint: use the equation for the trajectory.

Working: Substituting into $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$:

$$1 = 3\tan\alpha - \frac{g \times 3^2}{2 \times 3^2 \times \cos^2\alpha}$$

Simplifying gives $1 = 3\tan\alpha - \dfrac{g}{2\cos^2\alpha}$

E.g. 2 A golf ball is struck with an angle of 60^o and passes through the point $(4, 2)$. What was the initial speed of the ball?

Working: Substituting into $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$:

$$2 = 4\tan 60 - \frac{g \times 4^2}{2u^2\cos^2 60}$$

$2 = 4\sqrt{3} - \dfrac{8g}{\frac{1}{4}u^2}$ since $\tan 60 = \sqrt{3}$ and $\cos 60 = \dfrac{1}{2}$

$\dfrac{32g}{u^2} = 4\sqrt{3} - 2$ $\Rightarrow$ $u^2 = \dfrac{32g}{4\sqrt{3} - 2}$ $\Rightarrow$ $u = 7.98$

The initial speed of the ball 7.98 m/s (3 s.f.)

Video: Equation of trajectory
Video: Finding speed and direction

Exercise
p460 20B Qu 1-3, 4-6 (red)