Fixed-Point Iteration

Starter

1. Consider the equation $x^3 + x = 20$. Rearrange the equation so that $x = g(x)$. You should be able to find at least 3 different equations.

Working: Here are three possible rearrangements:

A. $x = 20 - x^3$

B. $x = \dfrac{20}{x^2 + 1}$

C. $x = (20 - x)^{\frac{1}{3}}$

E.g. 1 Solve $x^3 + x = 20$ to 5 d.p. using fixed-point iteration.
From the starter, we know that the equation can rearrange to

A. $x = 20 - x^3$ ***…or…***

B. $x = \dfrac{20}{x^2 + 1}$ ***…or…***

C. $x = (20 - x)^{\frac{1}{3}}$.

The solution to $x^3 + x = 20$ would satisfy each of the three equations.
At the moment we don't know which one to choose so we will look at each one individually.

A. $x = 20 - x^3$

The iterative formula is $x_{n+1} = 20 - x_n^3$.

Working:
We start with an $x-$value close to the root and this is our x_0.
Using the change of sign method we can find that the root lies between 2 and 3.
Therefore, we can choose $x_0 = 2.5$.

$x_1 = 20 - x_0^3 = 20 - 2.5^3 = 4.375$ ***that's not looking too clever***

$x_2 = 20 - x_1^3 = 20 - 4.375^3 = -63.74...$ ***even worse***

$x_3 = 20 - x_2^3 = 20 - (-63.74)^3 = -258984.9...$ ***I give up***

This is clearly not converging to the root so $x_{n+1} = 20 - x_n^3$ does not work

B. $x = \dfrac{20}{x^2 + 1}$

The iterative formula is $x_{n+1} = \dfrac{20}{x_n^2 + 1}$

Working:
Again let's start with $x_0 = 2.5$.

$x_1 = \dfrac{20}{x_0^2 + 1} = \dfrac{20}{2.5^2 + 1} = 2.758621$ (6 d.p.) ***looks promising***

$x_2 = \dfrac{20}{x_1^2 + 1} = \dfrac{20}{2.758621^2 + 1} = 2.322884$ (6 d.p.) ***getting there***

$x_3 = \dfrac{20}{x_2^2 + 1} = \dfrac{20}{2.322884^2 + 1} = 3.127058$ (6 d.p.) ***what?***

Pressing ANS on your calculator a few more times and you find out that the values are oscillating between 19.949874…and 0.050126 which is clearly no good for us.

c. $x = (20 - x)^{\frac{1}{3}}$

Working:
The iterative formula is $x_{n+1} = (20 - x_n)^{\frac{1}{3}}$
$x_0 = 2.5.$
$x_1 = (20 - x_0)^{\frac{1}{3}} = (20 - 2.5)^{\frac{1}{3}} = 2.596247$ *that's more like it*
$x_2 = (20 - x_1)^{\frac{1}{3}} = (20 - 2.596247)^{\frac{1}{3}} = 2.591479$ *already started to converge*
$x_3 = (20 - x_2)^{\frac{1}{3}} = (20 - 2.591479)^{\frac{1}{3}} = 2.591715$
Pressing ANS on your calculator a few more times and you can see that $x = 2.59170$ (5 d.p.)

E.g. 2 (a) Using the iterative formula $x_{n+1} = \sqrt{\frac{3x_n}{5}}$ and the starting value $x_0 = 1$, find the next three iterative values to 3 d.p.
(b) Find the equation to which this gives an approximate solution.

Working: (a) $x_0 = 1$
$$x_1 = \sqrt{\frac{3x_0}{5}} = \sqrt{\frac{3 \times 1}{5}} = 0.775$$
Now use the back arrow button and replace 1 by ANS, press "="
$x_2 = 0.682$
From now on you can just press "=" to obtain successive iterations
$x_3 = 0.640$

(b) Remove the n and $n + 1$ from the iterative formula and rearrange
$$x = \sqrt{\frac{3x}{5}}$$
$$x^2 = \frac{3x}{5}$$
$$5x^2 = 3x$$
$$5x^2 - 3x = 0$$

E.g. 3 (a) Show that $x^3 + 3x^2 - 7 = 0$ has a root in the interval between $x = 1$ and $x = 2$.
(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{7 - x_n{}^3}{3}}$ with $x_0 = 1$ to find values for x_1, x_2, x_3, x_4 to 3 d.p..
(c) Find the root to 3 d.p..

Working: (a) Let $f(x) = x^3 + 3x^2 - 7$
$f(1) = 1^3 + 3 \times 1^2 - 7 < 0$
$f(1) = 2^3 + 3 \times 2^2 - 7 > 0$
Since there is a sign change and the curve is continuous, there is a root between $x = 1$ and $x = 2$.

(b) $x_1 = \sqrt{\dfrac{7 - {x_0}^3}{3}} = \sqrt{\dfrac{7 - 1^3}{3}} = 1.414$

Back button, replace 1 with ANS, press "=": $x_2 = 1.179$ (3 d.p.)
Press "=": $x_3 = 1.337$ (3 d.p.)
Press "=": $x_4 = 1.240$ (3 d.p.)

(c) Keep pressing "=" until the successive values are equal
$x = 1.279$ (3 d.p.)

E.g. 4 An intersection of the curves $y = \ln x$ and $y = x - 2$ is at the point $x = a$ where $a = 3.1$ to 1 d.p. Use the iterative formula $x_{n+1} = 2 + \ln x_n$ to find the root to 3 decimal places.

Working:
$x_0 = 3.1$
$x_1 = 2 + \ln x_0 = 2 + \ln 3.1 = 3.1314$
$x_2 = 2 + \ln x_1 = 2 + \ln 3.1314 = 3.1415$
$x_3 = 3.1457$
$x_4 = 3.1457$
$x_5 = 3.1460$
$x_6 = 3.1461$
A few more presses confirms that the root is at 3.146 to 3 d.p.

E.g. 5 Let $f(x) = \ln 2x + x^3$.
(a) Show that $f(x)$ has a solution in the interval $0.4 < x < 0.5$.
(b) By setting $f(x) = 0$ and rearranging, find an iterative formula involving e^x. Hence solve $f(x) = 0$ to 3 d.p.

Working: (a) $f(0.4) = \ln 0.8 + 0.4^3 < 0$
$f(0.5) = \ln 1 + 0.5^3 > 0$
Since there is a sign change and the curve is continuous, there is a root in the interval $0.4 < x < 0.5$.

(b) $\ln 2x + x^3 = 0$
$\ln 2x = -x^3$
$2x = e^{-x^3}$
$x = \frac{1}{2}e^{-x^3}$ so the iterative formula is $x_{n+1} = \frac{1}{2}e^{-x_n^3}$
$x_0 = 0.45$
$x_1 = 0.4565$
$x_2 = 0.4546$
$x_3 = 0.4552$
So the root is 0.455 to 3 d.p.

Video: Fixed-point iteration

Fixed-point ilteration EQ

Solutions to Starter and E.g.s

Exercise
p316 14D Qu 1i, 2i, 4-7
(Make sure the your calculator is in radians when a questions involves trigonometry)