General binomial expansion

Starter

1. **(Review of last lesson)** Express $\dfrac{6x^2+11x-8}{(x+2)^2(x-1)}$ in partial fractions.

Working: $\dfrac{6x^2+11x-8}{(x+2)^2(x-1)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

Multiply by $(x+2)^2(x-1)$***:***

$6x^2+11x-8 \equiv A(x+2)^2 + B(x+2)(x-1) + C(x-1)$

Let $x=-2$: $\quad 24-22-8 = C(-2-1) \quad \therefore C = 2$

Let $x=1$: $\quad 6+11-8 = A(1+2)^2 \quad \therefore A = 1$

Let $x=0$: $\quad -8 = 4A - 2B - C \quad \therefore B = 5$

So $\dfrac{6x^2+11x-8}{(x+2)^2(x-1)} \equiv \dfrac{1}{x-1} + \dfrac{5}{x+2} + \dfrac{2}{(x+2)^2}$

Here is the formula for the binomial expansion that was covered in the AS course:

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 \ldots + {}^nC_{n-1} a b^{n-1} + b^n$$

2. **(Review of AS material)** Expand $(3-2x)^6$.

Working:

$$(3-2x)^5 = (2x)^5 + {}^5C_1.3^4.(-2x) + {}^5C_2.3^3.(-2x)^2 + {}^5C_3.3^2.(-2x)^3 + {}^5C_4.3.(-2x)^4 + (-2x)^5$$
$$= 243 + 5\times 81\times(-2x) + 10\times 27\times 4x^2 + 10\times 9\times(-8x^3) + 5\times 3\times 16x^4 - 32x^5$$
$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

E.g. 1 Expand the following up to and including the term in x^3:

(a) $(1+x)^{-2}$ (b) $(1+x)^{-5}$ (c) $\dfrac{1}{1+4x}$

Working:

(a)
$$(1+x)^{-2} = 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \ldots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \ldots$$

(b)
$$(1+x)^{-5} = 1 - 5x + \frac{(-5)(-6)}{2!}x^2 + \frac{(-5)(-6)(-7)}{3!}x^3 + \ldots$$
$$= 1 - 5x + 15x^2 - 35x^3 + \ldots$$

(c)
$$\frac{1}{1+4x} = (1+4x)^{-1}$$
$$(1+4x)^{-1} = 1 - (4x) + \frac{(-1)(-2)}{2!}(4x)^2 + \frac{(-1)(-2)(-3)}{3!}(4x)^3 + \ldots$$
$$= 1 - 4x + 16x^2 - 64x^3 + \ldots$$

E.g. 2 Expand the following up to and including the term in x^3:

(a) $(1+x)^{\frac{3}{4}}$ (b) $(1-x)^{\frac{3}{2}}$ (c) $\left(1-6x\right)^{\frac{4}{3}}$

Working: (a)
$$(1+x)^{\frac{3}{4}} = 1 + \frac{3}{4}x + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}x^3 + \dots$$
$$= 1 + \frac{3}{4}x - \frac{3}{32}x^2 + \frac{5}{128}x^3 + \dots$$

(b)
$$(1-x)^{\frac{3}{2}} = 1 + \frac{3}{2}(-x) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(-x)^2 + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(-x)^3 + \dots$$
$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \dots$$

(c)
$$\left(1-6x\right)^{\frac{4}{3}} = 1 + \frac{4}{3}(-6x) + \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)}{2!}(-6x)^2 + \frac{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{3!}(-6x)^3 + \dots$$
$$= 1 - 8x + 8x^2 + \frac{32}{3}x^3 + \dots$$

E.g. 3 State the interval of values of x for which these expansions are valid:

(a) $(1+2x)^{\frac{2}{3}}$ (b) $\left(1+\frac{3}{8}x\right)^{-5}$ (c) $\left(1-7x\right)^{-\frac{1}{4}}$

Working: (a) Since the power is not a positive integer, the expansion is valid for:
$-1 < 2x < 1$ i.e. $-\frac{1}{2} < x < \frac{1}{2}$

(b) Since the power is not a positive integer, the expansion is valid for:
$-1 < \frac{3}{8}x < 1$ i.e. $-\frac{8}{3} < x < \frac{8}{3}$

(c) Since the power is not a positive integer, the expansion is valid for:
$-1 < -7x < 1$

Dividing by a negative changes the direction of inequalities:

$\frac{1}{7} > x > -\frac{1}{7}$

Equality signs allows point to the left in compound inequalities:

$\therefore -\frac{1}{7} < x < \frac{1}{7}$

Exercise

p111 6A Qu 1iabc, 2, 4, 7, (10-11 red)

E.g. 4 Find the first four terms in the following expansions:

(a) $(4+x)^{\frac{1}{2}}$ (b) $\dfrac{1}{(3+x)^2}$

Working: (a) $(4+x)^{\frac{1}{2}} = \left[4\left(1+\frac{x}{4}\right)\right]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{4}\right)^3+\dots\right)$$
$$= 2\left(1+\frac{1}{8}x-\frac{1}{128}x^2+\frac{1}{1024}x^3+\dots\right)$$
$$= 2+\frac{1}{4}x-\frac{1}{64}x^2+\frac{1}{512}x^3+\dots$$

(b) $\dfrac{1}{(3+x)^2} = \dfrac{1}{\left[3\left(1+\frac{x}{3}\right)\right]^2} = \dfrac{1}{3^2\left(1+\frac{x}{3}\right)^2} = \dfrac{\left(1+\frac{x}{3}\right)^{-2}}{9} = \dfrac{1}{9}\left(1+\dfrac{x}{3}\right)^{-2}$

$$\frac{1}{9}\left(1+\frac{x}{3}\right)^{-2} = \frac{1}{9}\left(1-2\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{3}\right)^3+\dots\right)$$
$$= \frac{1}{9}\left(1-\frac{2}{3}x+\frac{1}{3}x^2-\frac{4}{27}x^3+\dots\right)$$
$$= \frac{1}{9}-\frac{2}{27}x+\frac{1}{27}x^2-\frac{4}{243}x^3+\dots$$

E.g. 5 You are given that $(1-3x)^{\frac{1}{5}} = 1-\frac{3}{5}x-\frac{18}{25}x^2+\dots$.

By substituting $x = \frac{1}{32}$, find an approximation for $\sqrt[5]{29}$ to four significant figures.

Hint: it is not simply a case of substituting $x = \frac{1}{32}$. Firstly, replace x by $\frac{1}{32}$ in $(1-3x)^{\frac{1}{5}}$ to see what value it gives.

Working: When $x = \frac{1}{32}$, $(1-3x)^{\frac{1}{5}} = \left(1-3\times\frac{1}{32}\right)^{\frac{1}{5}} = \sqrt[5]{\frac{29}{32}} = \frac{\sqrt[5]{29}}{2}$

So $\sqrt[5]{29} = 2\times(1-3x)^{\frac{1}{5}}$ when $x = \frac{1}{32}$

$$\sqrt[5]{29} \approx 2\left(1-\frac{3}{5}\times\frac{1}{32}-\frac{18}{25}\times\left(\frac{1}{32}\right)^2\right)$$
$= 1.961$ (4 s.f.)

E.g. 6 The first three terms of the expansion of $(1-2x)^{\frac{1}{2}}$ are $1-x-\frac{1}{2}x^2$. Using this expansion, find an approximation for $\sqrt{0.98}$.

Working: ***We are not given the $x-$ value to use so equate the expression to be expanded to what needs to be approximated.***

$(1-2x)^{\frac{1}{2}}=\sqrt{0.98} \quad \Rightarrow \quad 1-2x=0.98 \quad \therefore x=0.01$

So by replacing x by 0.01, the expansion will give the approximation.

Let $x=0.01$:

$$\sqrt{0.98} \approx 1-0.01-\frac{1}{2}\times 0.01^2$$
$$=1-0.01-0.00005$$
$$=0.98995$$

Video: **General binomial expansion (from 50:57)**

Binomial expansion - estimating a value EQ
Binomial expansion - rational and negative powers EQ

Solutions to Starter and E.g.s

Exercise
p111 6A Qu 1id, 3-6, 8, (9 red)