Harmonic identities

Starter

1. Find an approximation of $\dfrac{1-\cos 2\theta}{\theta \tan \theta}$ when θ is small.

Working: $$\frac{1-\cos 2\theta}{\theta \tan \theta} \approx \frac{1-(1-\frac{1}{2}(2\theta)^2)}{\theta \times \theta} = \frac{2\theta^2}{\theta^2} = 2$$

2. Find $\displaystyle\lim_{\theta \to 0} \frac{\sin 3\theta + \tan 5\theta}{2 \sin \theta}$.

Working: $$\lim_{\theta \to 0} \frac{\sin 3\theta + \tan 5\theta}{2 \sin \theta} = \frac{3\theta + 5\theta}{2\theta} = 4$$

3. Let $3 \sin \theta + 4 \cos \theta \equiv R \sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.
   (a) By using the compound angle identity for $\sin(A+B)$, find values for $R \sin \alpha$ and $R \cos \alpha$.
   (b) Using your answers to (a) find the values of R and α.

Working: (a)

$3 \sin \theta + 4 \cos \theta \equiv R \sin(\theta + \alpha)$

$R \sin(\theta + \alpha) = R(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$

$3 \sin \theta + 4 \cos \theta \equiv R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$

Equating coefficients:

$\sin \theta$: $R \cos \alpha = 3$

$\cos \theta$: $R \sin \alpha = 4$

(b)

Using the equations: $R \sin \alpha = 4$ and $R \cos \alpha = 3$

Squaring and adding: $R^2 \sin^2 \alpha + R^2 \cos^2 \alpha = 4^2 + 3^2$

$R^2(\sin^2 \alpha + \cos^2 \alpha) = 25$

Since $\sin^2 \alpha + \cos^2 \alpha = 1$: $R^2 = 25$

Since $R > 0$: $R = 5$

Divide the equations: $\dfrac{R \sin \alpha}{R \cos \alpha} = \dfrac{4}{3}$

$\tan \alpha = \dfrac{4}{3}$

$\alpha = \tan^{-1} \dfrac{4}{3} \approx 53.1°$

The value of R and α are 5 and $53.1°$ (3 s.f.) respectively.

4. Let $12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Using similar working to question 2, find the values of R and α.

Working:

$12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$
$R\cos(\theta + \alpha) = R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$
$12\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Equating coefficients:

$\cos\theta$: $R\cos\alpha = 12$

$\sin\theta$: $R\sin\alpha = 5$

Squaring and adding: $R^2\cos^2\alpha + R^2\sin^2\alpha = 12^2 + 5^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 169$

Since $\sin^2\alpha + \cos^2\alpha = 1$: $R^2 = 169$

Since $R > 0$: $R = 13$

Divide the equations: $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{5}{12}$

$\tan\alpha = \dfrac{5}{12}$

$\alpha = \tan^{-1}\dfrac{5}{12} \approx 22.6°$

The value of R and α are 13 and $22.6°$ (3 s.f.) respectively.

E.g. 1 (a) Given that $a\sin\theta + b\cos\theta \equiv R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, write down expressions for R and α in terms of a and b.

(b) Given that $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, write down expressions for R and α in terms of a and b.

Working:

(a) $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\dfrac{b}{a}$

(b) $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\dfrac{b}{a}$

E.g. 2 (a) Let $a\sin\theta - b\cos\theta \equiv R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Find expressions for R and α in terms of a and b.

(b) Hence write down expressions for R and α in terms of a and b given that $a\cos\theta - b\sin\theta \equiv R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

Working: (a)

$a\sin\theta - b\cos\theta \equiv R\sin(\theta - \alpha)$

$R\sin(\theta - \alpha) = R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$a\sin\theta - b\cos\theta \equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Equating coefficients:

$\sin\theta$: $R\cos\alpha = a$

$\cos\theta$: $R\sin\alpha = b$

Squaring and adding: $R^2\sin^2\alpha + R^2\cos^2\alpha = a^2 + b^2$

Since $\sin^2\alpha + \cos^2\alpha = 1$: $R^2 = a^2 + b^2$

Since $R > 0$: $R = \sqrt{a^2 + b^2}$

Divide the equations:

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a}$$

$$\tan\alpha = \frac{b}{a}$$

$$\alpha = \tan^{-1}\frac{b}{a}$$

(b) $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\dfrac{b}{a}$

E.g. 3 (a) Write $2\sin\theta - 9\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

(b) Write $7\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

Working: (a)

$R = \sqrt{2^2 + 9^2} = \sqrt{85}$ and $\alpha = \tan^{-1}\left|\dfrac{9}{2}\right| = 77.5°$

$2\sin\theta - 9\cos\theta \equiv \sqrt{85}\sin(\theta - 77.5°)$

(b)

$R = \sqrt{7^2 + 3^2} = \sqrt{58}$ and $\alpha = \tan^{-1}\left|\dfrac{3}{7}\right| = 23.2°$

$7\cos\theta - 3\sin\theta \equiv \sqrt{58}\cos(\theta + 23.2°)$

E.g. 4 (a) Solve the equation $\sin\theta + \sqrt{3}\cos\theta = 1$ for $0^\circ \le \theta \le 360^\circ$.

(b) Solve the equation $4\cos\theta + 9\sin\theta = 5$ for $0^c \le \theta \le 2\pi$.

Working: (a) Let $\sin\theta + \sqrt{3}\cos\theta \equiv R\sin(\theta + \alpha)$

$R = \sqrt{1^2 + 3} = 2$ and $\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60^\circ$

$\sin\theta + \sqrt{3}\cos\theta \equiv 2\sin(\theta + 60^\circ)$

$\sin\theta + \sqrt{3}\cos\theta = 1 \quad \Rightarrow \quad 2\sin(\theta + 60^\circ) = 1$

$\sin(\theta + 60^\circ) = \frac{1}{2}$

Since $\sin^{-1}\frac{1}{2} = 30^\circ$ and $\frac{1}{2}$ is positive:

$\theta + 60^\circ = 30^\circ$ ***(A quadrant)*** $\Rightarrow \theta = -30^\circ$

or $\theta + 60^\circ = 180^\circ - 30^\circ$ ***(S quadrant)*** $\Rightarrow \theta = 90^\circ$

Since -30° is not $0^\circ \le \theta \le 360^\circ$, add 360°.

So $\theta = 90^\circ$ and $\theta = 330^\circ$

(b) Let $4\cos\theta + 9\sin\theta \equiv R\cos(\theta - \alpha)$

$R = \sqrt{4^2 + 9^2} = \sqrt{97}$ and $\alpha = \tan^{-1}\left|\frac{9}{4}\right| = 1.1526^c$

$4\cos\theta + 9\sin\theta \equiv \sqrt{97}\cos(\theta - 1.1526^c)$

$4\cos\theta + 9\sin\theta = 5 \quad \Rightarrow \quad \sqrt{97}\cos(\theta - 1.1526^c) = 5$

$\cos(\theta - 1.1526^c) = \frac{5}{\sqrt{97}}$

Since $\cos^{-1}\frac{5}{\sqrt{97}} = 1.0383^c$ and $\frac{5}{\sqrt{97}}$ is positive:

$\theta - 1.1526^c = 1.0383^c$ ***(A quadrant)*** $\Rightarrow \theta = 2.19^c$

or $\theta - 1.1526^c = 2\pi - 1.0383^c$ ***(C quadrant)*** $\Rightarrow \theta = 6.40^c$

Since 6.40^c is not $0^c \le \theta \le 2\pi$, add 2π.

So $\theta = 0.114^c$ and $\theta = 2.19^c$.

E.g. 5 Find the range of values of the constant k for which the equation $\sin\theta + \sqrt{2}\cos\theta = k$ has real solutions for θ.

Working: Let $\sin\theta + \sqrt{2}\cos\theta \equiv R\sin(\theta - \alpha)$ then $R = \sqrt{1^2 + 2} = \sqrt{3}$.

$\sin\theta + \sqrt{2}\cos\theta = k \quad \Rightarrow \quad \sqrt{3}\sin(\theta + \alpha) = k$

$\sin(\theta + \alpha) = \frac{k}{\sqrt{3}}$

Since $-1 \le \sin x \le 1$: $\quad -1 \le \frac{k}{\sqrt{3}} \le 1$

Multiply by $\sqrt{3}$: $\quad -\sqrt{3} \le k \le \sqrt{3}$

E.g. 6 State the transformations that take the curve $y = \cos\theta$ to the curve $y = 8\cos\theta - 3\sin\theta$.

Working: Let $8\cos\theta - 3\sin\theta \equiv R\cos(\theta + \alpha)$

$R = \sqrt{8^2 + 3^2} = \sqrt{73}$ and $\alpha = \tan^{-1}\left|\frac{3}{8}\right| = 20.6^\circ$

$y = 8\cos\theta - 3\sin\theta \quad \Rightarrow \quad y = \sqrt{73}\cos(\theta + 20.6^\circ)$

The transformations are:

a stretch, factor $\sqrt{73}$, parallel to the $y-$axis.

a translation parallel to the $x-$axis, 20.6° (or 0.359^c) units to the left

E.g. 7 (a) Express $f(\theta) = 7\sin\theta - 24\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

(b) Hence find the greatest value of $\frac{9}{f(\theta) + 40}$. State the value of θ, where $0^\circ \le \theta \le 360^\circ$, for which this occurs.

Working: (a) $R = \sqrt{7^2 + 24^2} = 25$ and $\alpha = \tan^{-1}\left|\frac{24}{7}\right| = 73.7^\circ$

$f(\theta) = 25\sin(\theta - 73.7^\circ)$

(b) $\frac{9}{f(\theta) + 40} = \frac{9}{25\sin(\theta - 73.7^\circ) + 40}$

The greatest value is when the denominator is the least value.

Since $-1 \le \sin x \le 1$: $\quad -25 \le 25\sin(\theta - 73.7^\circ) \le 25$

Adding 40: $\quad 15 \le 25\sin(\theta - 73.7^\circ) + 40 \le 65$.

The greatest value is when the denominator is the least.

The greatest value of $\frac{9}{f(\theta) + 40}$ is $\frac{9}{-25 + 40} = \frac{9}{15} = \frac{3}{5}$.

This value occurs when $\sin(\theta - 73.7) = -1$

$\theta - 73.7 = 270^\circ$

$\theta = 344^\circ$ (3 s.f.)

Video: Harmonic identities

Harmonic identities EQ

Solutions to Starter and E.g.s

Exercise

p177 8C Qu 1-8, (9-10 red)