Hypothesis test for a sample mean

Starter

1. (a) If $X_1, X_2, X_3, \ldots, X_n$ is a random sample from N$(\mu, 1)$, state the distribution of the sample mean $\bar{X}$.

   (b) Find the least sample size required to ensure that the probability that $\bar{X}$ is within 0.1 of μ is greater than 0.95.

Working:

(a) $X \sim \text{N}(\mu, 1) \quad \Rightarrow \quad \bar{X}_n \sim \text{N}\left(\mu, \dfrac{1}{n}\right)$

(b) $P(|\bar{X} - \mu| < 0.1) = 0.95 \Rightarrow P(\mu - 0.1 < \bar{X} < \mu + 0.1) = 0.95$

$P(\bar{X} < \mu + 0.1) = 0.95 + \dfrac{0.05}{2} = 0.975$

Let z be the z−value corresponding to $\mu + 0.1$.

$P(Z < z) = 0.975 \quad \Rightarrow \quad z = 1.96$

Substitute into $z = \dfrac{x-\mu}{\frac{\sigma}{\sqrt{n}}}$: $\quad 1.96 = \dfrac{\mu + 0.1 - \mu}{\sqrt{\frac{1}{n}}}$

$1.96 = 0.1 \times \sqrt{n}$

$n \approx 384.1$

The sample size required is 385.

E.g. 1 Use the "p−value" method to test these hypotheses at the stated levels:

(a) $H_0 : \mu = 5, H_1 : \mu \neq 5, n = 25, \bar{x} = 6.1, \sigma = 3.0$ at the $5\,\%$ level

(a) $H_0 : \mu = 100, H_1 : \mu < 100, n = 36, \bar{x} = 98.5, \sigma = 5$ at the $5\,\%$ level

(a) $H_0 : \mu = 15, H_1 : \mu > 15, n = 40, \bar{x} = 16.5, \sigma = 3.5$ at the $1\,\%$ level

Working:

(a) $X \sim \text{N}(5, 3.0^2) \quad \Rightarrow \quad \bar{X}_{25} \sim \text{N}\left(5, \dfrac{3.0^2}{25}\right)$

Since $\bar{x} = 6.1$ is greater than $\mu = 5$, calculate $P(\bar{X} > 6.1)$.

$p = P(\bar{X} > 6.1) = 0.0334 \equiv 3.34\,\%$

Since it is a two-tailed test, $\alpha = \dfrac{5\,\%}{2} = 2.5\,\%$

Since $p = 3.34\,\% > 2.5\,\%$, we ***do not reject*** H_0 at the $5\,\%$ level.

(b) $X \sim \text{N}(100, 5^2) \quad \Rightarrow \quad \bar{X}_{36} \sim \text{N}\left(100, \dfrac{5^2}{36}\right)$

$p = P(\bar{X} < 98.5) = 0.0359 \equiv 3.59\,\%$

Since $p = 3.59\,\% < 5\,\%$, we ***reject*** H_0 at the $5\,\%$ level.

(c) $X \sim \text{N}(15, 3.5^2) \quad \Rightarrow \quad \bar{X}_{40} \sim \text{N}\left(15, \dfrac{3.5^2}{40}\right)$

$P(\bar{X} > 16.5) = 0.00336 \equiv 0.336\,\%$

Since $p = 0.336\,\% < 1\,\%$, we ***reject*** H_0 at the $1\,\%$ level.

E.g. 2 Use the “critical region” method to test these hypotheses at the stated levels:

(a) $H_0 : \mu = 120, H_1 : \mu < 120, n = 30, \bar{x} = 119, \sigma = 2.0$ at the $5\,\%$ level.

(b) $H_0 : \mu = 12.5, H_1 : \mu > 12.5, n = 25, \bar{x} = 12.9, \sigma = 1.5$ at the $1\,\%$ level.

(c) $H_0 : \mu = 0, H_1 : \mu \neq 0, n = 45, \bar{x} = 0.9, \sigma = 3.0$ at the $5\,\%$ level.

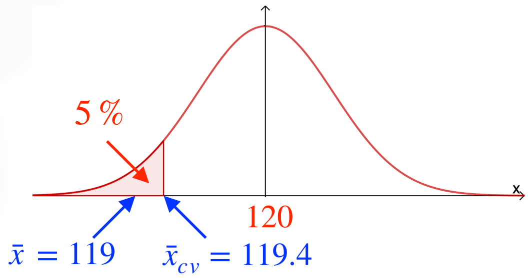
Draw a separate diagram for each question.

Working: (a) $X \sim \text{N}(120, 2.0^2) \quad \Rightarrow \quad \bar{X}_{30} \sim \text{N}\left(120, \frac{2.0^2}{30}\right)$

Let $\bar{x}_{cv}$ be the critical value such that $P(\bar{X} < \bar{x}_{cv}) = 0.05$

$\Rightarrow \bar{x}_{cv} = 119.4$

Since $\bar{x} = 119 < 119.4 = \bar{x}_{cv}$, the ***test statistic lies in the critical region*** so we ***reject*** H_0.

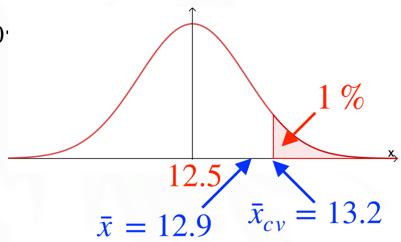


(b) $X \sim \text{N}(12.5, 1.5^2) \quad \Rightarrow \quad \bar{X}_{25} \sim \text{N}\left(12.5, \frac{1.5^2}{25}\right)$

Let $\bar{x}_{cv}$ be the critical value such that $P(\bar{X} > x_{cv}) = 0.05$.

$P(\bar{X} < x_{cv}) = 1 - 0.05 = 0.95 \quad \Rightarrow \quad x_{cv} = 13.2$

Since $\bar{x} = 12.9 < 13.2 = \bar{x}_{cv}$, the ***test statistic does not lie in the critical region*** so we ***do not reject*** H_0.

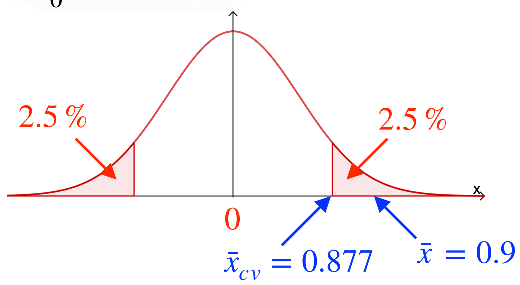


(c) $X \sim \text{N}(0, 3.0^2) \quad \Rightarrow \quad \bar{X}_{45} \sim \text{N}\left(0, \frac{3.0^2}{45}\right)$

Let $\bar{x}_{cv}$ be the critical value such that $P(\bar{X} > \bar{x}_{cv}) = 0.025$.

$P(\bar{X} < \bar{x}_{cv}) = 1 - 0.025 = 0.975 \Rightarrow \quad \bar{x}_{cv} = 0.877$

Since $\bar{x} = 0.9 > 0.877 = \bar{x}_{cv}$, the ***test statistic lies in the critical region*** so we ***reject*** H_0.



N.B. When you do worded questions make sure that you write H_0 and H_1 clearly.

E.g. 3 The IQ scores of a population are normal distributed with a mean of 100 and standard deviation of 15. A psychologist wishes to test the theory that eating chocolate before sitting an IQ test improves your score. A random sample of 81 people are selected and they are each given a 50 g bar of chocolate to eat before taking a standard IQ test. Their mean score on the test was 103.1. Test the psychologist's theory at the 2% level.

Working: $X \sim \text{N}(100, 15^2) \quad \Rightarrow \quad \bar{X}_{81} \sim \text{N}\left(100, \dfrac{15^2}{81}\right)$

The test statistic is $\bar{x} = 102.5$.
$H_0 : \mu = 100$
$H_1 : \mu > 100$
$p = P(\bar{X} > 103.1) = 0.0314 \equiv 3.14\%$
Since $p = 3.14\% > 2\%$, we ***do not reject*** H_0 at the 2% level.
There is evidence to suggest that eating chocolate before an IQ test does not improve performance.

E.g. 4 The mean weight of men in a particular country is found to have a standard deviation of 9.1 kg. The sample mean based on a random sample of 120 men was found to be 75.3 kg. A medical journal claims that the weight of men in this country are normally distributed with mean of 77.1 kg. Test the validity of this statement at the 5% level.

Working: $X \sim \text{N}(77.1, 9.1^2) \quad \Rightarrow \quad \bar{X}_{120} \sim \text{N}\left(77.1, \dfrac{9.1^2}{120}\right)$

The test statistic is $\bar{x} = 75.3$.
$H_0 : \mu = 77.1$
$H_1 : \mu < 77.1$
Let $\bar{x}_{cv}$ be the critical value such that $P(\bar{X} < \bar{x}_{cv}) = 0.05. \Rightarrow \quad \bar{x}_{cv} = 75.7$
Since $\bar{x} = 75.3 < 75.7 = \bar{x}_{cv}$, the ***test statistic lies in the critical region*** so we ***reject*** H_0.
There is no evidence to suggest that the mean weight of men in the country is 77.1 kg as claimed by the medical journal.

Video: An introduction to hypothesis testing
Video: Mean of Normal distribution hypothesis testing
Video: Sample means: hypothesis testing example

Solutions to Starter and E.g.s

Exercise
p405 18B Qu 1i, 2i, 3i, 4-8, (9 red)