Implicit Differentiation

Starter

1. **(Review of last lesson)**

For the curve $y = \frac{2x}{\cos x}$, find the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

Working: $\frac{dy}{dx} = = \frac{2\cos x + 2x \sin x}{\cos^2 x}$

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2\cos\frac{\pi}{3} + 2 \times \frac{\pi}{3} \sin\frac{\pi}{3}}{\cos^2 \frac{\pi}{3}}$

$$\frac{dy}{dx} = \frac{2 \times \frac{1}{2} + 2 \times \frac{\pi}{3} \times \frac{\sqrt{3}}{2}}{\frac{1}{4}}$$

$$\frac{dy}{dx} = 4 + \frac{4\pi\sqrt{3}}{3}$$

2. Find: (a) $\frac{d(x^2)}{dx}$ (b) $\frac{d(e^{7\cos 3x})}{dx}$ (c) $\frac{d(\ln \sin 3x)}{dx}$

Working: (a) $2x$

(b) $-21 \sin 3x e^{7\cos 3x}$

(c) $\frac{3\cos 3x}{\sin 3x}$

3. State the derivative of $y = \left[f(x)\right]^n$.

Working: $\frac{dy}{dx} = nf'(x) \times [f(x)]^{n-1}$

E.g. 1 Differentiate $y^2 = x$ with respect to x

Hint: Let $u = y^2$...

Working: Let $u = y^2$

Differentiating this equation w.r.t y gives $\frac{du}{dy} = 2y$

i.e. $\frac{dy}{du} = \frac{1}{2y}$

Replacing y^2 by u in the original equation gives $u = x$

So $\frac{du}{dx} = 1$

Substituting into the chain rule gives: $\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du} = 1 \times \frac{1}{2y}$

So $2y\frac{dy}{dx} = 1$

i.e. $\dfrac{d(y^2)}{dx} = 2y\dfrac{dy}{dx}$

E.g. 2 Differentiate $f(y) = x$ with respect to x

Working: Let $u = f(y)$ so $\dfrac{du}{dy} = f'(y)$ and $\dfrac{dy}{du} = \dfrac{1}{f'(y)}$

$u = x$ so $\dfrac{du}{dx} = 1$

Chain rule: $\dfrac{dy}{dx} = \dfrac{du}{dx} \times \dfrac{dy}{du} = 1 \times \dfrac{1}{f'(y)}$ so $f'(y)\dfrac{dy}{dx} = 1$

E.g. 3 Differentiate these functions with respect to x

(a) $7y^3$ (b) y^4 (c) e^y (d) $\sin 4y$

Working:

(a) $21y^2 \dots$ ***differentiate the function of y with respect to y***

$21y^2\dfrac{dy}{dx}$ ***multiply the result by $\dfrac{dy}{dx}$***

(b) $4y^3 \dots$ ***differentiate the function of y with respect to y***

$4y^3\dfrac{dy}{dx}$ ***multiply the result by $\dfrac{dy}{dx}$***

(c) $e^y \dots$ ***differentiate the function of y with respect to y***

$e^y\dfrac{dy}{dx}$ ***multiply the result by $\dfrac{dy}{dx}$***

(d) $4\cos 4y \dots$ ***differentiate the function of y with respect to y***

$4\cos 4y\dfrac{dy}{dx}$ ***multiply the result by $\dfrac{dy}{dx}$***

E.g. 4 Differentiate $y + 3y^2 - 7x = 9$.

Working:

$\dfrac{dy}{dx} + 6y\dfrac{dy}{dx} - 7 = 0$ ***differentiate each term individually***

$\dfrac{dy}{dx}\left(1 + 6y\right) - 7 = 0$ ***factorise out the $\dfrac{dy}{dx}$***

$\dfrac{dy}{dx} = \dfrac{7}{6y+1}$ ***rearrange***

E.g. 5 Find: (a) $\dfrac{d(x^2y)}{dx}$ (b) $\dfrac{d(4x^3\sin y)}{dx}$ (c) $\dfrac{d(e^{3y}\tan 5x)}{dx}$

Working: (a) $\dfrac{d(x^2y)}{dx} = 2xy + x^2\dfrac{dy}{dx}$ ***product rule***

(b) $\dfrac{d(4x^3 \sin y)}{dx} = 12x^2 \sin y + 4x^3 \cos y \dfrac{dy}{dx}$ *product rule*

$= 4x^2\left(3 \sin y + x \cos y \dfrac{dy}{dx}\right)$ *factorise*

(c) $\dfrac{d(e^{3y} \tan 5x)}{dx} = 3e^{3y}\dfrac{dy}{dx} \tan 5x + 5e^{3y} \sec^2 5x$ *product rule*

$= e^{3y}\left(3 \tan 5x \dfrac{dy}{dx} + 5 \sec^2 5x\right)$ *factorise*

E.g. 6 Find the coordinates of the stationary point(s) of the graph $y^2 = 18x^3 - 6xy$.

Working: $2y\dfrac{dy}{dx} = 54x^2 - 6y - 6x\dfrac{dy}{dx}$ *differentiate with respect to x*

$\dfrac{dy}{dx}(2y + 6x) = 2(27x^2 - 3y)$ *factorise*

$\dfrac{dy}{dx} = \dfrac{27x^2 - 3y}{y + 3x}$ *rearrange*

A stationary point occurs when $\dfrac{dy}{dx} = 0$ so $\dfrac{27x^2 - 3y}{y + 3x} = 0$

i.e. $27x^2 - 3y = 0$ so $y = 9x^2$ *numerator equals zero*

Substitute into the original equation: $81x^4 = 18x^3 - 54x^3$

$81x^4 + 36x^3 = 0$

$9x^3(9x + 4) = 0$

So $x = 0$ or $x = \dfrac{-4}{9}$

Coordinates are (0, 0) and $\left(-\dfrac{4}{9}, \dfrac{16}{9}\right)$ *Sub. into original equation*

Video: Derivatives of implicit functions

Derivatives of implicit functions EQ

Solutions to Starter and E.g.s

Exercise

p213 10D Qu 1i, 2iabd, 3, 5-10, 12*