Indefinite Integration by Substitution

Starter

1. (Review of last lesson) Find: (a) $\int \cos(2x+7)dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x dx$

Working: (a) $\frac{1}{2}\sin(2x+7)+c$

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x dx = \Big[\tan x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \tan\frac{\pi}{6} - \tan\frac{\pi}{4} = \frac{\sqrt{3}}{3} - 1 = \frac{\sqrt{3}-3}{3}$

2. (a) Use the method "Let $u = \ldots$" to find $\int (3x-2)^7 dx$.

(b) Use a similar method to (a), find:

(i) $\int x(x^2-5)^7 dx$ (ii) $\int x(x-5)^7 dx$

Working: (a) Let $u = 3x-2 \Rightarrow \frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$

$\int (3x-2)^7 dx = \int \frac{1}{3}u^7 du$ *replace $3x-2$ and dx*

$= \frac{1}{24}u^8 + c$ *integrate with respect to u*

$= \frac{1}{24}(3x-2)^8 + c$ *replace u by $3x-2$*

(b) (i) Let $u = x^2-5 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\int x(x^2-5)^7 dx = \int x\frac{1}{2x}u^7 du$ *replace x^2-5 and dx*

$= \int \frac{1}{2}u^7 du$ *cancel the x*

$= \frac{1}{16}u^8 + c$ *integrate with respect to u*

$= \frac{1}{16}(x^2-5)^8 + c$ *replace u by x^2-5*

(ii) Let $u = x - 5 \Rightarrow \dfrac{du}{dx} = 1 \Rightarrow dx = du$

$\int x(x-5)^7 dx = \int xu^7 du$ *replace $x - 5$ and dx*

The first part was easier but there is still an x and it needs to be replaced by a function of u before it can be integrated.

Rearranging $u = x - 5$ we get $x = u + 5$

$= \int (u+5)u^7 du$

$= \int (u^8 + 5u^7) du$ *expand the brackets*

$= \frac{1}{9}u^9 + \frac{5}{8}u^8 + c$ *integrate with respect to u*

$= \frac{1}{72}u^8(8u + 45) + c$

$= \frac{1}{72}(x-5)^8(8(x-5) + 45) + c$ *replace u*

$= \frac{1}{72}(8x+5)(x-5)^8 + c$

E.g. 1 Find $\int (x+1)(x+3)^5 dx$.

Working: Let $u = x + 3 \Rightarrow \dfrac{du}{dx} = 1 \Rightarrow dx = du$

$\int (x+1)(x+3)^5 dx = \int (x+1)u^5 du$ *replace $x + 3$ and dx*

$= \int (u - 3 + 1)u^5 du$ *since $x = u - 3$*

$= \int (u-2)u^5 du$

$= \int (u^6 - 2u^5) du$ *expand the brackets*

$= \frac{1}{7}u^7 - \frac{1}{3}u^6 + c$ *integrate with respect to u*

$= \frac{1}{21}u^6(3u - 7) + c$

$= \frac{1}{21}(x+3)^6(3(x+3) - 7) + c$ *replace u*

$= \frac{1}{21}(3x+2)(x+3)^6 + c$

E.g. 2 Use a suitable substitution to find $\int x\sqrt{x^2+1}\,dx$.

Working: Let $u = x^2 + 1$ so $\frac{du}{dx} = 2x$ and $dx = \frac{du}{2x}$

$\int x\sqrt{x^2+1}\,dx = \int x u^{\frac{1}{2}} \frac{du}{2x}$ replace x^2+1 by u and dx by $\frac{du}{2x}$

$= \frac{1}{2}\int u^{\frac{1}{2}} du$ *the x cancels top and bottom*

$= \frac{1}{3}u^{\frac{3}{2}} + c$ *integrate*

$= \frac{1}{3}\sqrt{(x^2+1)^3} + c$ replace u by x^2+1

Or:

Let $u^2 = x^2 + 1$ so $2u\frac{du}{dx} = 2x$ and $dx = \frac{u}{x}du$

$\int x\sqrt{x^2+1}\,dx = \int xu\frac{u}{x}du$

$= \int u^2 du$

$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}\sqrt{(x^2+1)^3} + c$

E.g. 3 Use a suitable substitution to integrate:

(a) $\int x(2x^2-1)^4 dx$ (c) $\int \frac{\sin x}{\cos^3 x}dx$

Working: (a) Let $u = 2x^2 - 1 \Rightarrow \frac{du}{dx} = 4x \Rightarrow dx = \frac{du}{4x}$

$\int x(2x^2-1)^4 dx = \int x\frac{u^4}{4x}du$ *replace $x+3$ and dx*

$= \int \frac{1}{4}u^4 du$ *cancel the x*

$= \frac{1}{20}u^5 + c$ *integrate with respect to u*

$= \frac{1}{20}(2x^2-1)^5 + c$ *replace u*

(b) Let $u = \cos x \Rightarrow \dfrac{du}{dx} = -\sin x \quad \Rightarrow \quad dx = \dfrac{du}{-\sin x}$

$$\int \frac{\sin x}{\cos^3 x} dx = \int \frac{\sin x}{-\sin x} \times \frac{1}{u^3} du \quad \textit{replace } \cos x \textit{ and } dx$$

$$= -\int \frac{1}{u^3} du \quad \textit{cancel the } \sin x$$

$$= -\int u^{-3} du \quad \textit{transform before integrating}$$

$$= \frac{u^{-2}}{2} + c \quad \textit{integrate with respect to } u$$

$$= \frac{1}{2u^2} + c$$

$$= \frac{1}{2\cos^2 x} + c \quad \textit{replace } u \textit{ by } \cos x$$

$$= \frac{1}{2}\sec^2 x + x \quad \textit{since } \frac{1}{\cos x} = \sec x$$

Video: Integration by substitution
Video: Integration by substitution involving square roots

Integration by substitution EQ

Solutions to Starter and E.g.s

Exercise
p229 11C Qu 1i, 2i, 3i

Summary
Let $u = \ldots$
Make sure all terms in x and dx are replaced by terms in u and du before integrating.