Infinite geometric series

Starter

1. (Review of last lesson) Colin leaves school and starts a job with a salary of £16,000 in the first year. He receives an annual increase of 4% per annum.
   (a) Assuming that he works for 40 years, calculate:
      (i) the amount he will earn in his 40th year
      (ii) the total amount that he will earn over the 40 year period.
   (b) How many full years will he need to work for his total earnings to exceed more than £1 million?

Working: (a) (i) $u_{40} = 16000 \times 1.04^{39} = £73861.86$

(ii) Since $r > 1$ use $S_n = \dfrac{a(r^n - 1)}{r - 1}$: $S_{40} = \dfrac{16000(1.04^{40} - 1)}{1.04 - 1}$

Amount earned is = £1520408.25

(b) $\dfrac{16000(1.04^n - 1)}{1.04 - 1} > 1000000$

$1.04^n > 3.5 \quad \Rightarrow \quad \ln 1.04^n > \ln 3.5 \quad \Rightarrow \quad n \ln 1.04 > \ln 3.5$

$\therefore n > \dfrac{\ln 3.5}{\ln 1.04} = 31.9$

It'll take 32 years to exceed £1m career earnings.

2. (Review of last lesson)

Find the value of N which makes the sum $\sum_{r=1}^{N} (2 \times 5^r)$ (GP) greater than 50000.

Working: $u_1 = 2 \times 5^1 = 10 \quad u_2 = 50 \quad u_3 = 250$

The series is geometric with $a = 10$ and $r = 5$

$\sum_{r=1}^{N} (2 \times 5^r) > 50000$ means $\dfrac{10(5^N - 1)}{5 - 1} > 50000$

$5^N > 20001 \quad \Rightarrow \quad \ln 5^N > \ln 20001 \quad \Rightarrow \quad N \ln 5 > \ln 20001$

$\therefore N > \dfrac{\ln 20001}{\ln 5} = 6.15$

The value of N is 7.

3. True of false: the sum of an infinite number of positive numbers is always infinite.

Working: False (periodic or converging) — see below.

4. What is the sum to an infinite number of terms of an arithmetic series?

Working: $\pm\infty$ (i.e. not interesting)

E.g. 1 Decide whether the geometric series $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\dots$ converges.

Working: $r=\frac{u_2}{u_1}=\frac{4}{9}\div\frac{2}{3}=\frac{4}{9}\times\frac{3}{2}=\frac{2}{3}$

Since $-1<r<1$, the series converges.

E.g. 2 Find the sum to infinity of 243 + 81 + 27 + …

Working: $a=243,\ r=\frac{81}{243}=\frac{1}{3}$

Using $S_\infty=\frac{a}{1-r}$ $\qquad S_\infty=\frac{243}{1-\frac{1}{3}}=\frac{243}{\frac{2}{3}}=364.5$

E.g. 3 The sum to infinity of a geometric series is 45 and the common ratio is $\frac{3}{5}$. Find the first term.

Working: Using $S_\infty=\frac{a}{1-r}$: $\qquad \frac{a}{1-\frac{3}{5}}=45 \quad\Rightarrow\quad \frac{a}{\frac{2}{5}}=45$

$a=18$

E.g. 4 The sum to infinity of a geometric series is 6. If the first term of the series is equal to the common ratio, find the value of the tenth term. Express your answer in index form.

Working: $S_\infty=6$ and $r=a$: $\qquad \frac{a}{1-a}=6 \quad\Rightarrow\quad a=6-6a$

$\therefore a=r=\frac{6}{7}$

$u_n=ar^{n-1}$: $\qquad u_{10}=\frac{6}{7}\times\frac{6}{7}^{10-1}=\left(\frac{6}{7}\right)^{10}$

E.g. 5 A ball is dropped from a height of 1 m onto a table top. It always rises to a height of 0.9 of the height from which it was dropped. How far does it travel in total until it stops bouncing?
Hint: consider downwards and upwards motion separately.

Working: Downwards motion: $1+0.9+0.9^2+0.9^3+\dots \Rightarrow \quad a=1,\ r=0.9$

Using $S_\infty=\frac{a}{1-r}$: $\qquad S_\infty=\frac{1}{1-0.9}=10$

Upwards motion: $0.9+0.9^2+0.9^3+\dots \quad\Rightarrow\quad a=0.9,\ r=0.9$

Using $S_\infty=\frac{a}{1-r}$: $\qquad S_\infty=\frac{0.9}{1-0.9}=9$

Total distance travelled = 19 m

E.g. 6* The 5th, 9th and 12 terms of an arithmetic series, with common difference of 3 are the 1st 3 terms of a geometric series. Show that the geometric series is convergent and find its sum to infinity.

Working: AP with $d = 3$: $u_5 = a + (5-1) \times 3 = a + 12$

$u_9 = a + (9-1) \times 3 = a + 24$

$u_{12} = a + (12-1) \times 3 = a + 33$

GP: $a + 12, a + 24, a + 33$

$r = \dfrac{u_2}{u_1} = \dfrac{u_3}{u_2}$: $\dfrac{a+24}{a+12} = \dfrac{a+33}{a+24}$

Cross-multiply: $(a+24)^2 = (a+12)(a+33)$

$a^2 + 48a + 576 = a^2 + 45a + 396$

$3a = -180$

$a = -60$

When $a = -60$, $r = \dfrac{-60+24}{-60+12} = \dfrac{-36}{-48} = \dfrac{3}{4}$

Since $-1 < r < 1$, the geometric series is convergent.

The first term of the GP is the 5th term of AP: $-60 + 4 \times 3 = -48$

Using $S_\infty = \dfrac{a}{1-r}$: $S_\infty = \dfrac{-48}{1-\frac{3}{4}} = -192$

Video: Sum to infinity

Solutions to Starter and E.g.s

Exercise
p84 4G Qu 1i, 2i, 3-11