Integrals leading to Exponentials and Logs

Notes

1. **(Review of last lesson)** Find $\displaystyle\int_2^{2.5} \frac{1}{(4x-7)^2}\,dx.$

Working:

$$\int_2^{2.5} \frac{1}{(4x-7)^2}\,dx = \int_2^{2.5} (4x-7)^{-2}\,dx$$

$$= \left[\frac{1}{4\times(-1)}(4x-7)^{-1}\right]_2^{2.5}$$

$$= \frac{1}{4}\left[\frac{1}{(4x-7)}\right]_{2.5}^{2}$$

Remember: change the sign of integral, change the limits around.

$$= \frac{1}{4}\left[\frac{1}{(4\times 2-7)} - \frac{1}{(4\times 2.5-7)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{1} - \frac{1}{3}\right]$$

$$= \frac{1}{6}$$

2. Using the method of "Let $u = \ldots$", find (a) $\displaystyle\int e^{2x+1}\,dx$ (b) $\displaystyle\int \frac{1}{2x+1}\,dx.$

Working: (a) Let $u = 2x+1 \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow \dfrac{du}{2} = dx$

$$\int e^{2x+1}\,dx = \int \frac{1}{2}e^u\,du$$ ***replace $2x+1$ and dx***

$$= \frac{1}{2}e^u + c$$ ***integrate with respect to u***

$$= \frac{1}{2}e^{2x+1} + c$$ ***replace u by $2x+1$***

(b) Let $u = 2x+1 \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow \dfrac{du}{2} = dx$

$$\int \frac{1}{2x+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$ ***replace $2x+1$ and dx***

$$= \frac{1}{2}\ln u + c$$ ***integrate with respect to u***

$$= \frac{1}{2}\ln(2x+1) + c$$ ***replace u by $2x+1$***

3. Find $\int ke^{ax+b}dx$.

Working: Let $u = ax + b \quad \Rightarrow \quad \frac{du}{dx} = a \quad \Rightarrow \quad \frac{du}{a} = dx$

$$\int ke^{ax+b}dx = \int \frac{k}{a}e^u du \text{ or } \frac{k}{a}\int e^u du$$ ***replace*** $ax + b$ ***and*** dx

$$= \frac{k}{a}e^u + c$$ ***integrate with respect to*** u

$$= \frac{k}{a}e^{ax+b} + c$$ ***replace*** u ***by*** $ax + b$

E.g. 1 Find: (a) $\int e^{8x+1}dx$ (b) $\int \frac{1}{6x-5}dx$

Working: (a) $\frac{1}{8}e^{8x+1} + c$

(b) $\frac{1}{6}\ln(6x - 5) + c$

E.g. 2 Find: (a) $\int \frac{1}{e^{5x+3}}dx$ (b) $\int_2^3 e^{2x}dx$

Working: (a)

$$\int \frac{1}{e^{5x+3}}dx = \int e^{-5x-3}dx$$

$$= -\frac{1}{5}e^{-5x-3} + c$$

$$= -\frac{1}{5e^{5x+3}} + c$$

(b)

$$\int_2^3 e^{2x}dx = \frac{1}{2}\left[e^{2x}\right]_2^3$$

$$= \frac{1}{2}(e^6 - e^4)$$

E.g. 3 Express $\int_b^a \frac{15}{5+3x}dx$ as a single logarithm.

Working:

$$\int_b^a \frac{15}{5+3x}dx = 5\left[\ln\left|5+3x\right|\right]_b^a$$

$$= 5\left[\ln\left|5+3a\right| - \ln\left|5+3b\right|\right]$$

$$= 5\ln\left|\frac{5+3a}{5+3b}\right|$$ ***2nd law of logs***

$$= \ln\left|\frac{5+3a}{5+3b}\right|^5$$ ***3rd law of logs***

E.g. 4 The graph of the curve $y = f(x)$ passes through $(1, 2)$. Find $f(x)$ if $f'(x) = \dfrac{4}{10 - 9x}$.

Working: $$f(x) = \int \frac{4}{10 - 9x} dx = -\frac{4}{9} \ln \left| 10 - 9x \right| + c$$

Substitute $(1, 2)$: $2 = -\dfrac{4}{9} \ln 1 + c \quad \Rightarrow \quad c = 2$ ***since*** $\ln 1 = 0$

$$f(x) = -\frac{4}{9} \ln \left| 10 - 9x \right| + 2$$

E.g. 5 Given that $\displaystyle\int_1^A \frac{4}{6x - 5} dx = 10$ and $A \geq 1$, find A in terms of e.

Working: $$\int_1^A \frac{4}{6x - 5} dx = \frac{4}{6} \Big[\ln \left| 6x - 5 \right| \Big]_1^A$$

$$= \frac{2}{3} \Big[\ln \left| 6A - 5 \right| - \ln \left| 6 - 5 \right| \Big]$$

$$= \frac{2}{3} \ln \left| 6A - 5 \right|$$ ***since*** $\ln 1 = 0$

$$\frac{2}{3} \ln \left| 6A - 5 \right| = 10 \quad \Rightarrow \quad \ln \left| 6A - 5 \right| = 15$$

$$6A - 5 = e^{15} \quad \Rightarrow \quad A = \frac{e^{15} + 5}{6}$$

Video: Integration by inspection (e^x)
Video: Integrals leading to ln

Solutions to Starter and E.g.s

Exercise
p222 11A Qu 2i, 3i