Integrals of the form $\dfrac{f'(x)}{f(x)}$

Starter

1. **(Review of previous A2 material)** Differentiate:

(a) $y = \ln(3x^2 + 5x)$ (b) $y = \ln(\sin x)$ (c) $y = \ln(5e^x + 2x)$

Hint: For $y = k\ln[f(x)] \Rightarrow \dfrac{dy}{dx} = \dfrac{kf'(x)}{f(x)}$

Working: Using $y = k\ln[f(x)] \Rightarrow \dfrac{dy}{dx} = \dfrac{kf'(x)}{f(x)}$

(a) $\dfrac{dy}{dx} = \dfrac{6x + 5}{3x^2 + 5x}$

(b) $\dfrac{dy}{dx} = \dfrac{\cos x}{\sin x} = \cot x$

(c) $\dfrac{dy}{dx} = \dfrac{5e^x + 2}{5e^x + 2x}$

2. Using your answers to question 1, integrate the following:

(a) $\displaystyle\int \frac{10x + 7}{5x^2 + 7x}dx$ (b) $\displaystyle\int \frac{\cos x}{4 + \sin x}dx$ (c) $\displaystyle\int \frac{e^x}{3e^x - 1}dx$

Working: (a) $\ln\left|5x^2 + 7x\right| + c$

(b) $\ln(4 + \sin x) + c$

(c) $\dfrac{1}{3}\ln\left|3e^x - 1\right| + c$

E.g. 1 Find: (a) $\displaystyle\int \frac{x^3}{1 + x^4}dx$ (b) $\displaystyle\int \frac{x - 1}{3x^2 - 6x + 1}dx$

Working: (a) $\dfrac{d}{dx}\ln\left|1 + x^4\right| = \dfrac{4x^3}{1 + x^4}$ so we need to divide by 4

$\displaystyle\int \frac{x^3}{1 + x^4}dx = \frac{1}{4}\ln\left|1 + x^4\right| + c$

(b) $\displaystyle\int \frac{x - 1}{3x^2 - 6x + 1}dx = \frac{1}{6}\ln\left|3x^2 - 6x + 1\right| + c$

E.g. 2 Since x^3 is a multiple of the derivative of $1 + x^4$, we can find $\int \frac{x^3}{1+x^4} dx$ using $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$.

Decide whether these adjustments mean that $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$?

Give your answer as "still works" or "does not work anymore".

(a) $\int \frac{5x^3}{1+x^4} dx$ (b) $\int \frac{x^2}{1+x^4} dx$ (c) $\int \frac{x^3}{1+x^3} dx$

(d) $\int \frac{x^3}{6+x^4} dx$ (e) $\int \frac{x^3}{1+7x^4} dx$

Working: (a) Still works

(b) Does not work anymore because the derivative of x^4 is $4x^3$ which is not a multiple of x^2

(c) Does not work anymore because the derivative of x^3 is $3x^2$ which is not a multiple of x^3

(d) Still works

(e) Still works

E.g. 3 Find: (a) $\int \frac{2x+3}{x^2+3x-4} dx$ (b) $\int \tan x \, dx$

Hint: For (b), remember the identity involving sine, cosine and tan.

Working: (a) $\ln|x^2+3x-4| + c$

(b) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + c = \ln|\sec x| + c$

E.g. 4 Find: (a) $\int \frac{dx}{5x+2}$ (b) $\int \frac{dx}{1-2x}$ (c) $\int \frac{x^3}{3x^4-5} dx$

Working: (a) $\frac{1}{5}\ln|5x+2| + c$

(b) $-\frac{1}{2}\ln|1-2x| + c$

(c) $\frac{1}{12}\ln|3x^4-5| + c$

E.g. 5 Find $\int_0^1 \frac{x}{x^2+1}dx$, expressing your answer in the form $\ln\sqrt{a}$ where a is an integer.

Working:

$$\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\left[\ln\left|x^2+1\right|\right]_0^1$$
$$= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2 = \ln\sqrt{2}$$

E.g. 6* Find $\int \frac{1}{x\ln x}dx$.

Hint $\int \frac{1}{x\ln x}dx \equiv \int \frac{\frac{1}{x}}{\ln x}dx$

Working: $\int \frac{1}{x\ln x}dx \equiv \int \frac{\frac{1}{x}}{\ln x}dx = \ln\left|\ln x\right| + c$

E.g. 7* Which adjustment(s) need to be made to $\int \frac{4e^x - 15x^3}{4e^{2x} - 5x^6}dx$ so that $\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$ ***can*** be used?

Working: ***Either*** change e^x to e^{2x} and change x^3 to x^5

or change e^{2x} to e^x and change $5x^6$ to $\frac{15}{4}x^4$.

Video: Integrals of the form f'(x)/f(x)

Solutions to Starter and E.g.s

Exercise
C3/C4 book (purple) p174 Ex 2 Qu 1-12, 13, 14 ace…, 15, 16ac, 17ace