# Integrating Parametric Equations

**Starter**

1. (Review of last lesson) Find $\frac{dy}{dx}$ for the curve $x = \ln t$, $y = 3t^2 - t^3$.

***Working:*** $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = 6t - 3t^2 = 3t(2-t)$

Using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$: $\frac{dy}{dx} = \frac{3t(2-t)}{\frac{1}{t}}$

$\frac{dy}{dx} = 3t^2(2-t)$

2. (Review of A2 material) Find $\int_{\frac{\pi}{2}}^{\pi} x \sin x \, dx$.

***Working:*** Integration by parts

Let $u = x \Rightarrow u' = 1$

Let $v' = \sin x \Rightarrow v = -\cos x$

Using $\int uv' = uv - \int u'v$:

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} x \sin x \, dx &= \Big[x \times -\cos x\Big]_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} 1 \times -\cos x \, dx \\ &= \Big[-x\cos x\Big]_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} \cos x \, dx \\ &= \Big[-x\cos x + \sin x\Big]_{\frac{\pi}{2}}^{\pi} \\ &= \Big(-\pi\cos\pi + \sin\pi\Big) - \Big(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\Big) \\ &= \pi - 1\end{aligned}$$

***E.g. 1*** Find an expression in parametric form that is equivalent to $\int y \, dx$:

(a) $x = \frac{3}{t}$, $y = 4t^2$ (b) $x = \sqrt{t}$, $y = 3t^2 - 4$

***Working:*** (a) From $x = \frac{3}{t} = 3^{-t}$, $\frac{dx}{dt} = -3t^{-2} = -\frac{3}{t^2}$

$$\int y\frac{dx}{dt}dt = \int 4t^2 \times -\frac{3}{t^2}dt = -\int 12 dt$$

(b) From $x = \sqrt{t} = t^{\frac{1}{2}}$, $\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

$$\int y\frac{dx}{dt}dt = = \int 3t^2 - 4 \times \frac{1}{2\sqrt{t}}dt = \int \frac{3t^2-4}{2\sqrt{t}}dt$$

***E.g. 2*** A curve has parametric equations $x = 3t^2$, $y = \dfrac{5}{t}$, where $t > 0$. Find the value of $\displaystyle\int_3^{75} y\,dx$.

***Working:***

When $x = 75$, $3t^2 = 75 \Rightarrow t^2 = 25$
$\therefore t = 5$ since $t > 0$

From $x = 3t^2$, $\dfrac{dx}{dt} = 6t$

$$\int_3^{75} y\frac{dx}{dt}dt = \int_1^5 \frac{5}{t} \times 6t\,dt$$
$$= \int_1^5 30\,dt = 120$$
$$= \Big[30t\Big]_1^5$$
$$= 30 \times 5 - 30 \times 1$$
$$= 120$$

***E.g. 3*** A curve has parametric equations $x = 6t^2$, $y = e^{2t}$, where $t > 0$. Find the value of $\displaystyle\int_0^6 y\,dx$.

***Working:***

When $x = 6$, $t = 1$ (since $t > 0$)
When $x = 0$, $t = 0$

$$\int_0^6 y\,dx = \int_0^1 (e^{2t} \times 12t)\,dt = \int_0^1 12te^{2t}\,dt$$

Let $u = 12t \Rightarrow u' = 12$

Let $v' = e^{2t} \Rightarrow v = \dfrac{1}{2}e^{2t}$

$$\int_0^1 12te^{2t}\,dt = \Big[6te^{2t}\Big]_0^1 - \int_0^1 6e^{2t}\,dt$$
$$= \Big[6te^{2t} - 3e^{2t}\Big]_0^1$$
$$= \left(6e^2 - 3e^2\right) - \left(0 - 3e^0\right)$$
$$= 3e^2 + 3$$
$$= 3(e^2 + 1)$$

**Video A:** Integrating parametric functions
**Video B:** Integrating parametric functions

**Solutions to Starter and E.g.s**

**Exercise**
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