

Integration by Parts

Starter

1. **(Review of last lesson)** Using a suitable substitution express $\int_1^4 \frac{1}{3 - \sqrt{x}} dx$ in the form $a + b \ln 2$ where a and b are integers.

Working: Let $u = 3 - \sqrt{x} = 3 - x^{\frac{1}{2}} \Rightarrow \frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}} \Rightarrow -2x^{\frac{1}{2}} du = dx$

Table of changed values:

When $x = 4, u = 3 - \sqrt{4} = 1$

When $x = 1, u = 3 - \sqrt{1} = 2$

x	u
4	1
1	2

$$\begin{aligned} \int_1^4 \frac{1}{(1 + e^x)^2} dx &= \int_2^1 \frac{-2x^{\frac{1}{2}}}{u} du && \text{replace } 1 + e^x, dx \text{ and limits} \\ &= \int_1^2 \frac{2(3 - u)}{u} du^{***} && \text{since } x^{\frac{1}{2}} = 3 - u \\ &= \int_1^2 \left(\frac{6}{u} - 2 \right) du && \text{transform before integrating} \\ &= \left[6 \ln u - 2u \right]_1^2 && \text{integrate with respect to } u \\ &= (6 \ln 2 - 2 \times 2) - (6 \ln 1 - 2 \times 1) && \text{substitute} \\ &= -2 + 6 \ln 2 \end{aligned}$$

N.B. ***Change the limits around, change the sign of the integral.

2. **(Review of A2 material)** Differentiate: (a) $x \sin x$ (b) $3xe^{5x}$.

Working: (a) $\frac{d(x \sin x)}{dx} = \sin x + x \cos x$

(b) $\frac{d(3xe^{5x})}{dx} = 3e^{5x} + 15xe^{5x} = 3e^{5x}(1 + 5x)$

E.g. 1 Find $\int 15xe^{5x} dx$ by using the same method, including the answer from the starter.

Working:

$$\begin{aligned} \int 15xe^{5x} dx &= \int \left(\frac{d(3xe^{5x})}{dx} - 3e^{5x} \right) dx \\ &= 3xe^{5x} - \int 3e^{5x} dx \\ &= 3xe^{5x} - \frac{3}{5}e^{5x} + c \end{aligned}$$

E.g. 2 Find $\int xe^x dx$.

Working: **Choice 1:** Let $u = e^x \Rightarrow u' = e^x$
 Let $v' = x \Rightarrow v = \frac{1}{2}x^2$

Using $\int uv' = uv - \int u'v$:

$$\int xe^x dx = xe^x - \int \frac{1}{2}x^2 e^x dx$$

The function to be integrated has become more complicated so this must be the wrong choice.

Choice 2: Let $u = x \Rightarrow u' = 1$
 Let $v' = e^x \Rightarrow v = e^x$

Using $\int uv' = uv - \int u'v$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

How do we decide which function is u and which is v' ?

v' is usually the **more complicated function** but we **need to be able to integrate v'** .

E.g. 3 Find: (a) $\int x \sin x dx$ (b) $\int x(1+x)^7 dx$ (c) $\int \ln x dx$

Working: (a) Let $u = x \Rightarrow u' = 1$
 Let $v' = \sin x \Rightarrow v = -\cos x$

Using $\int uv' = uv - \int u'v$:

$$\int x \sin x dx = -x \cos x + \int \cos x dx = \sin x - x \cos x + c$$

(b) Let $u = x \Rightarrow u' = 1$
 Let $v' = (1+x)^7 \Rightarrow v = \frac{1}{8}(1+x)^8$

Using $\int uv' = uv - \int u'v$:

$$\begin{aligned} \int x(1+x)^7 dx &= x \times \frac{1}{8}(1+x)^8 - \int \frac{1}{8}(1+x)^8 dx \\ &= \frac{x}{8}(1+x)^8 - \frac{1}{72}(1+x)^9 + c \\ &= \frac{1}{72}(1+x)^8(9x - (1+x)) + c \\ &= \frac{1}{72}(8x - 1)(x + 1)^8 + c \end{aligned}$$

- (c) **N.B.** This is the exception when v' is not the more complicated function.

$$\text{Let } u = \ln x \Rightarrow u' = \frac{1}{x}$$

$$\text{Let } v' = 1 \Rightarrow v = x$$

$$\text{Using } \int uv' = uv - \int u'v:$$

$$\begin{aligned} \int \ln x dx &= \ln x \times x - \int \frac{1}{x} \times x dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c \end{aligned}$$

N.B. So $v' \neq \ln x$.

E.g. 4 Find: (a) $\int_1^2 x^5 \ln x dx$ (b) $\int_1^2 x\sqrt{x-1} dx$ (c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x(1 - \sin x) dx$

Working: (a) Let $u = \ln x \Rightarrow u' = \frac{1}{x}$
Let $v' = x^5 \Rightarrow v = \frac{x^6}{6}$

$$\text{Using } \int uv' = uv - \int u'v:$$

$$\begin{aligned} \int_1^2 x^5 \ln x dx &= \left[\ln x \times \frac{x^6}{6} \right]_1^2 - \int_1^2 \frac{1}{x} \times \frac{x^6}{6} dx \\ &= \left[\frac{x^6}{6} \ln x \right]_1^2 - \int_1^2 \frac{x^5}{6} dx \\ &= \left[\frac{x^6}{6} \ln x - \frac{x^6}{36} \right]_1^2 \\ &= \left(\frac{2^6}{6} \ln 2 - \frac{2^6}{36} \right) - \left(\frac{1^6}{6} \ln 1 - \frac{1^6}{36} \right) \\ &= \frac{32}{3} \ln 2 - \frac{7}{4} \end{aligned}$$

(b) Let $u = x \Rightarrow u' = 1$
Let $v' = \sqrt{x-1} \Rightarrow v = \frac{2(x-1)^{\frac{3}{2}}}{3}$

Using $\int uv' = uv - \int u'v$:

$$\begin{aligned}\int_1^2 x\sqrt{x-1} dx &= \left[x \times \frac{2(x-1)^{\frac{3}{2}}}{3} \right]_1^2 - \int_1^2 1 \times \frac{2(x-1)^{\frac{3}{2}}}{3} dx \\ &= \left[\frac{2x(x-1)^{\frac{3}{2}}}{3} - \frac{4(x-1)^{\frac{5}{2}}}{15} \right]_1^2 \\ &= \left(\frac{2 \times 2(2-1)^{\frac{3}{2}}}{3} - \frac{4(2-1)^{\frac{5}{2}}}{15} \right) - (0-0) \\ &= \left(\frac{4}{3} - \frac{4}{15} \right) \\ &= \frac{16}{15}\end{aligned}$$

(c) Let $u = 2x \Rightarrow u' = 2$
Let $v' = 1 - \sin x \Rightarrow v = x + \cos x$

Using $\int uv' = uv - \int u'v$:

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x(1 - \sin x) dx &= \left[2x(x + \cos x) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(x + \cos x) dx \\ &= \left[2x^2 + 2x \cos x - x^2 - \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \left[x^2 + 2x \cos x - \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \left(\frac{\pi^2}{2} + 2 \times \frac{\pi}{2} \cos \frac{\pi}{2} - \sin \frac{\pi}{2} \right) - \left(\left(-\frac{\pi}{2} \right)^2 + 2 \times \left(-\frac{\pi}{2} \right) \cos \left(-\frac{\pi}{2} \right) - \sin \left(-\frac{\pi}{2} \right) \right) \\ &= -4\end{aligned}$$

[Video: Integration by parts](#)

[Video: Integration by parts involving In](#)

[Video: Integration by parts with limits](#)

[Integration by parts EQ](#)

[Solutions to Starter and E.g.s](#)

Exercise

p232 11D Qu 1-3