Integration by Parts

Starter

1. **(Review of last lesson)** Using a suitable substitution express $\int_1^4 \frac{1}{3-\sqrt{x}}dx$ in the form $a + b\ln 2$ where a and b are integers.

Working: Let $u = 3 - \sqrt{x} = 3 - x^{\frac{1}{2}} \Rightarrow \frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}} \Rightarrow -2x^{\frac{1}{2}}du = dx$

Table of changed values:

When $x = 4, u = 3 - \sqrt{4} = 1$

When $x = 1, u = 3 - \sqrt{1} = 2$

x	u
4	1
1	2

$$\int_1^4 \frac{1}{3-\sqrt{x}}dx = \int_2^1 \frac{-2x^{\frac{1}{2}}}{u}du$$ ***replace $1+e^x$, dx and limits***

$$= \int_1^2 \frac{2(3-u)}{u}du \quad ***$$ ***since $x^{\frac{1}{2}} = 3-u$***

$$= \int_1^2 \left(\frac{6}{u} - 2\right)du$$ ***transform before integrating***

$$= \Big[6\ln u - 2u\Big]_1^2$$ ***integrate with respect to u***

$$= (6\ln 2 - 2\times 2) - (6\ln 1 - 2\times 1)$$ ***substitute***

$$= -2 + 6\ln 2$$

N.B. ***Change the limits around, change the sign of the integral.

2. **(Review of A2 material)** Differentiate: (a) $x\sin x$ (b) $3xe^{5x}$.

Working: (a) $\frac{d(x\sin x)}{dx} = \sin x + x\cos x$

(b) $\frac{d(3xe^{5x})}{dx} = 3e^{5x} + 15xe^{5x} = 3e^{5x}(1+5x)$

E.g. 1 Find $\int 15xe^{5x}dx$ by using the same method, including the answer from the starter.

Working:

$$\int 15xe^{5x}dx = \int\left(\frac{d(3xe^{5x})}{dx} - 3e^{5x}\right)dx$$

$$= 3xe^{5x} - \int 3e^{5x}dx$$

$$= 3xe^{5x} - \frac{3}{5}e^{5x} + c$$

E.g. 2 Find $\int xe^x dx$.

Working: **Choice 1:** Let $u = e^x \Rightarrow u' = e^x$

Let $v' = x \Rightarrow v = \frac{1}{2}x^2$

Using $\int uv' = uv - \int u'v$:

$$\int xe^x dx = xe^x - \int \frac{1}{2}x^2 e^x dx$$

The function to be integrated has become more complicated so this must be the wrong choice.

Choice 2: Let $u = x \Rightarrow u' = 1$

Let $v' = e^x \Rightarrow v = e^x$

Using $\int uv' = uv - \int u'v$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

How do we decide which function is u and which is v'?

v' is usually the ***more complicated function*** but we ***need to be able to integrate v'***.

E.g. 3 Find: (a) $\int x \sin x dx$ (b) $\int x(1+x)^7 dx$ (c) $\int \ln x dx$

Working: (a) Let $u = x \Rightarrow u' = 1$

Let $v' = \sin x \Rightarrow v = -\cos x$

Using $\int uv' = uv - \int u'v$:

$$\int x \sin x dx = -x\cos x + \int \cos x dx = \sin x - x\cos x + c$$

(b) Let $u = x \Rightarrow u' = 1$

Let $v' = (1+x)^7 \Rightarrow v = \frac{1}{8}(1+x)^8$

Using $\int uv' = uv - \int u'v$:

$$\begin{aligned}\int x(1+x)^7 dx &= x \times \frac{1}{8}(1+x)^8 - \int \frac{1}{8}(1+x)^8 dx \\ &= \frac{x}{8}(1+x)^8 - \frac{1}{72}(1+x)^9 + c \\ &= \frac{1}{72}(1+x)^8\big(9x - (1+x)\big) + c \\ &= \frac{1}{72}(8x-1)(x+1)^8 + c\end{aligned}$$

(c) ***N.B.*** This is the exception when v' is not the more complicated function.

Let $u = \ln x \quad \Rightarrow \quad u' = \dfrac{1}{x}$

Let $v' = 1 \quad \Rightarrow \quad v = x$

Using $\displaystyle\int uv' = uv - \int u'v$:

$$\int \ln x\,dx = \ln x \times x - \int \frac{1}{x} \times x\,dx$$
$$= x\ln x - \int 1\,dx$$
$$= x\ln x - x + c$$

N.B. So $v' \neq \ln x$.

E.g. 4 Find: (a) $\displaystyle\int_1^2 x^5 \ln x\,dx$ (b) $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$ (c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x(1-\sin x)\,dx$

Working: (a) Let $u = \ln x \quad \Rightarrow \quad u' = \dfrac{1}{x}$

Let $v' = x^5 \quad \Rightarrow \quad v = \dfrac{x^6}{6}$

Using $\displaystyle\int uv' = uv - \int u'v$:

$$\int_1^2 x^5 \ln x\,dx = \left[\ln x \times \frac{x^6}{6}\right]_1^2 - \int_1^2 \frac{1}{x} \times \frac{x^6}{6}\,dx$$
$$= \left[\frac{x^6}{6}\ln x\right]_1^2 - \int_1^2 \frac{x^5}{6}\,dx$$
$$= \left[\frac{x^6}{6}\ln x - \frac{x^6}{36}\right]_1^2$$
$$= \left(\frac{2^6}{6}\ln 2 - \frac{2^6}{36}\right) - \left(\frac{1^6}{6}\ln 1 - \frac{1^6}{36}\right)$$
$$= \frac{32}{3}\ln 2 - \frac{7}{4}$$

(b) Let $u = x \Rightarrow u' = 1$

Let $v' = \sqrt{x-1} \Rightarrow v = \dfrac{2(x-1)^{\frac{3}{2}}}{3}$

Using $\displaystyle\int uv' = uv - \int u'v$:

$$\int_1^2 x\sqrt{x-1}dx = \left[x \times \frac{2(x-1)^{\frac{3}{2}}}{3}\right]_1^2 - \int_1^2 1 \times \frac{2(x-1)^{\frac{3}{2}}}{3}dx$$

$$= \left[\frac{2x(x-1)^{\frac{3}{2}}}{3} - \frac{4(x-1)^{\frac{5}{2}}}{15}\right]_1^2$$

$$= \left(\frac{2 \times 2(2-1)^{\frac{3}{2}}}{3} - \frac{4(2-1)^{\frac{5}{2}}}{15}\right) - (0-0)$$

$$= \left(\frac{4}{3} - \frac{4}{15}\right)$$

$$= \frac{16}{15}$$

(c) Let $u = 2x \Rightarrow u' = 2$

Let $v' = 1 - \sin x \Rightarrow v = x + \cos x$

Using $\displaystyle\int uv' = uv - \int u'v$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x(1-\sin x)dx = \left[2x(x+\cos x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(x+\cos x)dx$$

$$= \left[2x^2 + 2x\cos x - x^2 - 2\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left[x^2 + 2x\cos x - 2\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2}^2 + 2 \times \frac{\pi}{2}\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2}\right) - \left(\left(-\frac{\pi}{2}\right)^2 + 2 \times \left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) - 2\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= -4$$

Video: Integration by parts
Video: Integration by parts involving ln
Video: Integration by parts with limits

Integration by parts EQ

Solutions to Starter and E.g.s

Exercise
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