Integration using Partial Fractions

Starter

1. **(Review of last lesson)** Find: (a) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 4x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x \sin 2x\,dx$

Working: (a)

$$\cos 2x \equiv 2\cos^2 x - 1 \quad \Rightarrow \quad \cos 8x \equiv 2\cos^2 4x - 1$$

$$\cos^2 4x \equiv \frac{1}{2}(1 + \cos 8x) \equiv \frac{1}{2} + \frac{1}{2}\cos 8x$$

$$\int_0^{\frac{\pi}{4}} \cos^2 4x\,dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 8x\right) dx$$

$$= \left[\frac{1}{2}x + \frac{1}{16}\sin 8x\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{8} + \frac{1}{16}\sin 2\pi\right) - \left(0 + \frac{1}{16}\sin 0\right)$$

$$= \frac{\pi}{8} = 0.393$$

(b)

$$2\sin x \cos x \equiv \sin 2x \quad \Rightarrow \quad 2\sin 2x \cos 2x \equiv \sin 4x$$

$$\therefore \sin 2x \cos 2x \equiv \frac{1}{2}\sin 4x$$

$$\int_0^{\frac{\pi}{4}} \cos 2x \sin 2x\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin 4x\,dx$$

$$= \left[-\frac{1}{8}\cos 4x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{1}{8}\cos 4x\right]_{\frac{\pi}{4}}^{0}$$

$$= \left(\frac{1}{8}\cos 0\right) - \left(\frac{1}{8}\cos \pi\right)$$

$$= \frac{1}{8}$$

2. **(Review of A2 material)** (a) Express $\dfrac{3x-2}{x^2+x-12}$ as partial fractions.

(b) Hence find $\displaystyle\int \frac{3x-2}{x^2+x-12}\,dx$, expressing your answer as a single logarithm.

Working: (a)

$$\frac{3x-2}{x^2+x-12} = \frac{3x-2}{(x+4)(x-3)} \equiv \frac{A}{x+4} + \frac{B}{x-3}$$

$$3x - 2 \equiv A(x-3) + B(x+4)$$

When $x = 3$: $\quad 7 = 7B \quad \Rightarrow \quad B = 1$

When $x = -4$: $\quad -14 = -7A \quad \Rightarrow \quad A = 2$

$$\frac{3x-2}{x^2+x-12} \equiv \frac{2}{x+4} + \frac{1}{x-3}$$

(b) $$\int \frac{3x-2}{x^2+x-12}dx = \int \frac{2}{x+4} + \frac{1}{x-3}dx$$
$$= 2\ln|x+4| + \ln|x-3| + c$$
$$= \ln|(x-3)(x+4)^2| + c$$

3. **(Review of AS material)**
Find the quotient and remainder when you divide $4x^3 - 2x^2 + 6$ by $x + 3$.

Working: $4x^3 - 2x^2 + 6 \equiv (x+3)(ax^2+bx+c) + d$ where $d \equiv$ remainder

Equating coefficients:

x^3: $4 = a$

x^2: $-2 = 3a + b$

Since $4 = a$, $b = -14$

x: $0 = c + 3b$

Since $b = -14$, $c = 42$

constant: $6 = 3c + d$

Since $c = 42$, $d = -120$

Quotient $= 4x^2 - 14x + 42$, Remainder $= -120$

N.B. The method of polynomial division is also possible.

E.g. 1 (a) Express $\dfrac{11x-10}{x(x-5)^2}$ as partial fractions.

(b) Hence find $\displaystyle\int \frac{11x-10}{x(x-5)^2}dx$

Working: (a) $$\frac{11x-10}{x(x-5)^2} \equiv \frac{A}{x} + \frac{B}{x-5} + \frac{C}{(x-5)^2}$$
$$11x - 10 \equiv A(x-5)^2 + Bx(x-5) + Cx$$

$x = 0$: $-10 = 25A \Rightarrow A = -\dfrac{2}{5}$

$x = 5$: $45 = 5C \Rightarrow C = 9$

$x = 10$: $100 = 25A + 250B + 10C \Rightarrow B = \dfrac{2}{5}$

$$\frac{11x-10}{x(x-5)^2} \equiv -\frac{2}{5x} + \frac{2}{5(x-5)} + \frac{9}{(x-5)^2}$$

(b) $$\int \frac{11x-10}{x(x-5)^2}dx = \int -\frac{2}{5x} + \frac{2}{5(x-5)} + \frac{9}{(x-5)^2}dx$$
$$= \int -\frac{2}{5x} + \frac{2}{5(x-5)} + 9(x-5)^{-2}dx$$
$$= -\frac{2}{5}\ln\left|x\right| + \frac{2}{5}\ln\left|x-5\right| - 9(x-5)^{-1} + c$$
$$= \frac{2}{5}\ln\left|\frac{x-5}{x}\right| - \frac{9}{(x-5)} + c$$ ***2nd law of logs***

E.g. 2 Evaluate $\int_0^1 \frac{x}{(x-2)(x-3)}dx$, expressing your answer as a single logarithm.

Working:

$$\frac{x}{(x-2)(x-3)} \equiv \frac{A}{x-2} + \frac{B}{x-3}$$
$$x \equiv A(x-3) + B(x-2)$$

When $x = 3$: $\quad 3 = B$

When $x = 2$: $\quad 2 = -A \quad \Rightarrow \quad A = -2$

$$\int_0^1 \frac{x}{(x-2)(x-3)}dx = \int_0^1 \frac{3}{x-3} - \frac{2}{x-2}dx$$
$$= \left[3\ln\left|x-3\right| - 2\ln\left|x-2\right|\right]_0^1$$
$$= \left(3\ln 2 - 2\ln 1\right) - \left(3\ln 3 - 2\ln 2\right)$$
$$= \ln 8 - 0 - \ln 27 + \ln 4$$
$$= \ln\frac{32}{27}$$

Video: Integrals involving partial fractions

Integrals involving partial fractions EQ

Video: Integration methods

Solutions to Starter and E.g.s

Exercise
p242 11G Qu 3i, 4i, 5i, 6-9