

Integration using Partial Fractions

Starter

1. **(Review of last lesson)** Find: (a) $\int_0^{\frac{\pi}{4}} \cos^2 4x dx$ (b) $\int_0^{\frac{\pi}{4}} \cos 2x \sin 2x dx$

Working: (a) $\cos 2x \equiv 2 \cos^2 x - 1 \Rightarrow \cos 8x \equiv 2 \cos^2 4x - 1$
 $\cos^2 4x \equiv \frac{1}{2}(1 + \cos 8x) \equiv \frac{1}{2} + \frac{1}{2} \cos 8x$
 $\int_0^{\frac{\pi}{4}} \cos^2 4x dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos 8x \right) dx$
 $= \left[\frac{1}{2}x + \frac{1}{16} \sin 8x \right]_0^{\frac{\pi}{4}}$
 $= \left(\frac{\pi}{8} + \frac{1}{16} \sin 2\pi \right) - \left(0 + \frac{1}{16} \sin 0 \right)$
 $= \frac{\pi}{8} = 0.393$

(b) $2 \sin x \cos x \equiv \sin 2x \Rightarrow 2 \sin 2x \cos 2x \equiv \sin 4x$
 $\therefore \sin 2x \cos 2x \equiv \frac{1}{2} \sin 4x$
 $\int_0^{\frac{\pi}{4}} \cos 2x \sin 2x dx = \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin 4x dx$
 $= \left[-\frac{1}{8} \cos 4x \right]_0^{\frac{\pi}{4}}$
 $= \left[\frac{1}{8} \cos 4x \right]_0^{\frac{\pi}{4}}$
 $= \left(\frac{1}{8} \cos 0 \right) - \left(\frac{1}{8} \cos \pi \right)$
 $= \frac{1}{4}$

2. **(Review of A2 material)** (a) Express $\frac{3x - 2}{x^2 + x - 12}$ as partial fractions.
 (b) Hence find $\int \frac{3x - 2}{x^2 + x - 12} dx$, expressing your answer as a single logarithm.

Working: (a) $\frac{3x - 2}{x^2 + x - 12} = \frac{3x - 2}{(x + 4)(x - 3)} \equiv \frac{A}{x + 4} + \frac{B}{x - 3}$
 $3x - 2 \equiv A(x - 3) + B(x + 4)$
 When $x = 3$: $7 = 7B \Rightarrow B = 1$
 When $x = -4$: $-14 = -7A \Rightarrow A = 2$
 $\frac{3x - 2}{x^2 + x - 12} \equiv \frac{2}{x + 4} + \frac{1}{x - 3}$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{3x-2}{x^2+x-12} dx &= \int \frac{2}{x+4} + \frac{1}{x-3} dx \\
 &= 2 \ln|x+4| + \ln|x-3| + c \\
 &= \ln|(x-3)(x+4)^2| + c
 \end{aligned}$$

3. **(Review of AS material)**

Find the quotient and remainder when you divide $4x^3 - 2x^2 + 6$ by $x + 3$.

Working: $4x^3 - 2x^2 + 6 \equiv (x + 3)(ax^2 + bx + c) + d$ where $d \equiv$ remainder

Equating coefficients:

$$\begin{aligned}
 x^3: & 4 = a \\
 x^2: & -2 = 3a + b \\
 & \text{Since } 4 = a, b = -14 \\
 x: & 0 = c + 3b \\
 & \text{Since } b = -14, c = 42 \\
 \text{constant:} & 6 = 3c + d \\
 & \text{Since } c = 42, d = -120
 \end{aligned}$$

Quotient = $4x^2 - 14x + 42$, Remainder = -120

N.B. The method of polynomial division is also possible.

E.g. 1 (a) Express $\frac{11x-10}{x(x-5)^2}$ as partial fractions.

(b) Hence find $\int \frac{11x-10}{x(x-5)^2} dx$

Working:

$$\begin{aligned}
 \text{(a)} \quad \frac{11x-10}{x(x-5)^2} &\equiv \frac{A}{x} + \frac{B}{x-5} + \frac{C}{(x-5)^2} \\
 11x-10 &\equiv A(x-5)^2 + Bx(x-5) + Cx \\
 x=0: & -10 = 25A \Rightarrow A = -\frac{2}{5} \\
 x=5: & 45 = 5C \Rightarrow C = 9 \\
 x=10: & 100 = 25A + 250B + 10C \Rightarrow B = \frac{2}{5} \\
 \frac{11x-10}{x(x-5)^2} &\equiv -\frac{2}{5x} + \frac{2}{5(x-5)} + \frac{9}{(x-5)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{11x-10}{x(x-5)^2} dx &= \int -\frac{2}{5x} + \frac{2}{5(x-5)} + \frac{9}{(x-5)^2} dx \\
 &= \int -\frac{2}{5x} + \frac{2}{5(x-5)} + 9(x-5)^{-2} dx \\
 &= -\frac{2}{5} \ln|x| + \frac{2}{5} \ln|x-5| - 9(x-5)^{-1} + c \\
 &= \frac{2}{5} \ln \left| \frac{x-5}{x} \right| - \frac{9}{(x-5)} + c \quad \text{2nd law of logs}
 \end{aligned}$$

E.g. 2 Evaluate $\int_0^1 \frac{x}{(x-2)(x-3)} dx$, expressing your answer as a single logarithm.

Working:

$$\frac{x}{(x-2)(x-3)} \equiv \frac{A}{x-2} + \frac{B}{x-3}$$
$$x \equiv A(x-3) + B(x-2)$$

When $x = 3$: $3 = B$

When $x = 2$: $2 = -A \Rightarrow A = -2$

$$\int_0^1 \frac{x}{(x-2)(x-3)} dx = \int_0^1 \frac{3}{x-3} - \frac{2}{x-2} dx$$
$$= \left[3 \ln|x-3| - 2 \ln|x-2| \right]_0^1$$
$$= (3 \ln 2 - 2 \ln 1) - (3 \ln 3 - 2 \ln 2)$$
$$= \ln 8 - 0 - \ln 27 + \ln 4$$
$$= \ln \frac{32}{27}$$

Video: [Integrals involving partial fractions](#)

Integrals involving partial fractions EQ

Video: [Integration methods](#)

[Solutions to Starter and E.g.s](#)

Exercise

p242 11G Qu 3i, 4i, 5i, 6-9