Introduction to normal probabilities

Starter
Download and print the starter activity **here**.

1. The heights of 1000 Hong Kong 18 year olds was measured to the nearest cm and the data is collated in the table below.

Height, h	Frequency	Lower bound	Upper bound	Class width	Frequency density
158–159	1				0.5
160–161	4				
162–163	22				
164–165	35				
166–167	79				
168–169	119				
170–171	131				
172–173	176				
174–175	133				
176–177	117				
178–179	83				
180–181	61				
182–183	22				
184–185	14				
186–187	1				
188–189	2				

(a) Explain why frequency density of the first group is 0.5.
(b) Complete the table.
(c) Draw a histogram using the axes on the **worksheet**.

Hint: Frequency density $= \dfrac{\text{Frequency}}{\text{Class width}}$

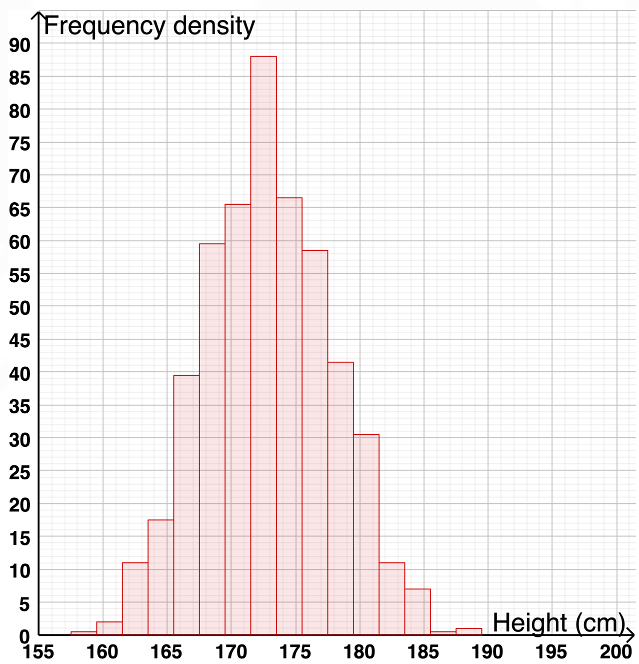
Working: (a) The class width is 2 since the lower bound is 157.5 and the upper bound is 159.5.

$$\text{Frequency density} = \frac{\text{Frequency}}{\text{Class width}} = \frac{1}{2} = 0.5$$

(b)

Height, h	Frequency	Lower bound	Upper bound	Class width	Frequency density
158–159	1	157.5	159.5	2	0.5
160–161	4	159.5	161.5	2	2
162–163	22	161.5	163.5	2	11
164–165	35	163.5	165.5	2	17.5
166–167	79	165.5	167.5	2	39.5
168–169	119	167.5	169.5	2	59.5
170–171	131	169.5	171.5	2	62.5
172–173	176	171.5	173.5	2	88
174–175	133	173.5	175.5	2	66.5
176–177	117	175.5	177.5	2	58.5
178–179	83	177.5	179.5	2	41.5
180–181	61	179.5	181.5	2	30.5
182–183	22	181.5	183.5	2	11
184–185	14	183.5	185.5	2	7
186–187	1	185.5	187.5	2	0.5
188–189	2	187.5	189.5	2	1

(c)



E.g. 1 For $X \sim \mathrm{N}(8,\,6)$, use your calculator to find:

(a) $P(9 < X \leq 10)$
(b) $P(X > 7)$
(c) $P(X \leq 6)$
(d) $P(X < 6.5,\, X \geq 9.1)$

Give your answers to 4 d.p..

Working:

(a) $P(9 < X \leq 10) = 0.1344$ (4 d.p.)

(b) $P(X > 7) = 0.6585$ (4 d.p.)

(c) $P(X \leq 6) = 0.2071$ (4 d.p.)

(d) $P(X < 6.5,\, X \geq 9.1) = 1 - P(6.5 < X \leq 9.1)$
$= 1 - 0.4032$
$= 0.5968$ (4 d.p.)

…or…

$P(X < 6.5,\, X \geq 9.1) = P(X < 6.5) + P(X \geq 9.1)$
$= 0.270146 + 0.326689$ (6 d.p.)
$= 0.5968$ (4 d.p.)

Video (Classwiz): **Finding probabilities for the normal distribution**

E.g. 2 For $X \sim \mathrm{N}(\mu,\,\sigma^2)$, state the values of:

(a) $P(X < \mu + \sigma)$
(b) $P(X < \mu - 2\sigma,\, X \geq \mu + 2\sigma)$
(c) $P(X \geq \mu + 2\sigma)$
(d) $P(\mu - \sigma < X < \mu + 2\sigma)$

Give your answers to 2 d.p..

Working:

(a) $P(X < \mu + \sigma) \approx 0.68 + \dfrac{1 - 0.68}{2} = 0.84$

(b) $P(X < \mu - 2\sigma,\, X \geq \mu + 2\sigma) \approx 1 - 0.95 = 0.05$

(c) $P(X \geq \mu + 2\sigma) \approx \dfrac{1 - 0.95}{2} = 2.5$

(d) $P(\mu - \sigma < X < \mu + 2\sigma) \approx 1 - P(X < \mu - \sigma) - P(X > \mu + 2\sigma)$
$= 1 - 0.16 - 0.025$
$= 0.815$

Location of the points of inflexion on the normal curve

E.g. 3 The equation of the normal curve with mean of zero and standard deviation σ could be written as $f(x) = ke^{-\frac{x^2}{2\sigma^2}}$. Find the $x-$coordinates of the points of inflexion.

Working: A point of inflexion occurs when $f''(x) = 0$.

$$f'(x) = -\frac{2kx}{2\sigma^2}e^{-\frac{x^2}{2\sigma^2}} = -\frac{k}{\sigma^2}xe^{-\frac{x^2}{2\sigma^2}}$$

Now use the product rule:

$u = x \quad \Rightarrow \quad u' = 1 \qquad -\frac{k}{\sigma^2}$ can come out as a factor

$v = e^{-\frac{x^2}{2\sigma^2}} \quad \Rightarrow \quad v' = -\frac{1}{\sigma^2}xe^{-\frac{x^2}{2\sigma^2}}$

$$f''(x) = -\frac{k}{\sigma^2}\left(1 \times e^{-\frac{x^2}{2\sigma^2}} + x \times -\frac{1}{\sigma^2}xe^{-\frac{x^2}{2\sigma^2}}\right)$$

$$= -\frac{k}{\sigma^2}\left(e^{-\frac{x^2}{2\sigma^2}} - \frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\right)$$

$$= -\frac{k}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left(1 - \frac{x^2}{\sigma^2}\right)$$

$f''(x) = 0$ when $\quad -\frac{k}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left(1 - \frac{x^2}{\sigma^2}\right) = 0$

$$1 - \frac{x^2}{\sigma^2} = 0 \quad \Rightarrow \quad 1 = \frac{x^2}{\sigma^2} \quad \Rightarrow \quad x^2 = \sigma^2 \quad \Rightarrow \quad x = \pm\sigma$$

The points of inflexion of the normal curve occur when $x = \pm\sigma$ i.e. at the standard deviations.

Video: Normal distribution

Video: Normal cumulative distribution

Solutions to Starter and E.g.s

Exercise

p381 17A Qu 1i, 2-4