Inverse Normal distribution

Starter

1. The heights of a group of people are normally distributed with a mean of 174 cm and a standard deviation of 6 cm. Find the probability that a person selected at random:
   (a) is at least 170 cm tall
   (b) is no taller than 180 cm
   (c) is at least 178 cm given that they are between 172 and 182 cm.

Working: (a) $X \sim \mathrm{N}(174,\, 6^2)$

$P(X \geq 170) = 0.7475$

The probability the person is at least 170 cm tall is 0.7475 (4 s.f.)

(b) $P(X \leq 180) = 0.8413$

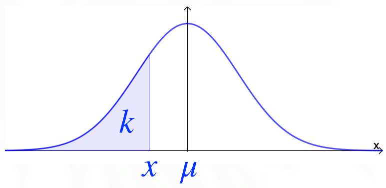
The probability the person is no taller than 180 cm is 0.8413 (4 s.f.)

(c)
$$\begin{aligned} P(X \geq 178 \mid 172 < X < 182) &= \frac{P(X \geq 178 \cap 172 < X < 182)}{P(172 < X < 182)} \\ &= \frac{P(178 \leq X < 182)}{P(172 < X < 182)} \\ &\approx \frac{0.16128}{0.53935} \\ &= 0.2990 \end{aligned}$$

The probability is at least 178 cm given that they are between 172 and 182 cm is 0.2990 (4 s.f.)

Considering areas diagrammatically

E.g. 1 The inverse normal function of a modern calculator calculates the $x-$value when the area is similar to the one in the diagram $P(X < x) = k$.

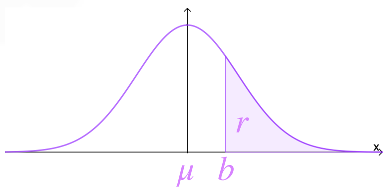


(a) Sketch a diagram and shade the area represented by r for $P(X > b) = r$. Write down a formula for $P(X < b)$ involving r.
(b) Sketch a diagram and shade the area represented by s for $P(|X - \mu| < b) = s$. Write down a formula for $P(X < \mu + b)$ involving s, but not $P(X < \mu - b)$.
(c) Sketch a diagram and shade the area represented by t for $P(a < X < b) = t$, where $a < \mu < b$. If $P(X < a)$ is given, find a formula for $P(X < b)$ in terms of t and $P(X < a)$.

Working: (a) $P(X > b) = r$

$P(X < b) = 1 - r$

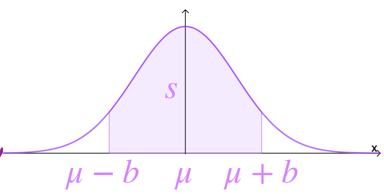
since area under normal curve is 1



(b) $P(|X - \mu| < b) = s$

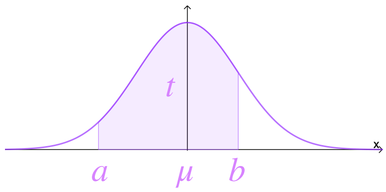
$P(\mu - b < X < \mu + b) = s$

$P(\mu < X < \mu + b) = \dfrac{s}{2}$ ***symmetry***



$P(X < \mu + b) = \dfrac{s}{2} + 0.5$ ***since*** $P(X < \mu) = 0.5$

(c) $P(a < X < b) = t$
$P(X < b) = t + P(X < a)$



E.g. 2 Given that $X \sim$ N$(52, 40)$, find p, q, r and s correct to 3 s.f. such that:

(a) $P(X < p) = 0.647$
(b) $P(X > q) = 0.581$
(c) $P(52 < X < r) = 0.3$
(d) $P(|X - 52| < s) = 0.36$

Working:

(a) $P(X < p) = 0.647 \quad \Rightarrow \quad p = 54.4$ (3 s.f.)

(b) $P(X > q) = 0.581 \quad \Rightarrow \quad P(X < q) = 1 - 0.581 = 0.419$
$\therefore q = 50.7$ (3 s.f.)

(c) $P(52 < X < r) = 0.3$
Since $\mu = 52$, $P(X < 52) = 0.5 \quad \Rightarrow \quad P(X < r) = 0.8$
$\therefore r = 57.3$ (3 s.f.)

(d) $P(|X - 52| < s) = 0.36$ is the same as
$P(52 - s < X < 52 + s) = 0.36$ ***symmetry about the mean***
$P(52 < X < 52 + s) = 0.18$
$\Rightarrow \quad P(X < 52 + s) = 0.68$ ***since*** $P(X < 52) = 0.5$
$52 + s = 54.958 \quad \Rightarrow \quad s = 2.96$ (3 s.f.)

E.g. 3 Given that $y \sim$ N$(20, 2^2)$, find a, b, c and d correct to 3 s.f. such that:

(a) $P(X > a) = 0.28$
(b) $P(X < b) = 0.183$
(c) $P(|X - 20| < c) = 0.84$
(d) $P(18 < X < d) = 0.45$

Working:

(a) $P(X > a) = 0.28 \quad \Rightarrow \quad P(X < a) = 1 - 0.28 = 0.72$
$\therefore a = 21.2$ (3 s.f.)

(b) $P(X < b) = 0.183 \quad \Rightarrow \quad b = 18.2$ (3 s.f.)

(c) $P(|X - 20| < c) = 0.84$ is the same as
$P(20 - c < X < 20 + c) = 0.84$ ***symmetry about the mean***
$P(20 < X < 20 + c) = 0.42$
$\Rightarrow \quad P(X < 20 + c) = 0.92$ ***since*** $P(X < 20) = 0.5$
$20 + c = 22.81 \quad \Rightarrow \quad c = 2.81$ (3 s.f.)

(d) $P(18 < X < d) = 0.45$
$P(X < 18) = 0.158655 \quad \Rightarrow \quad P(X < d) = 0.608655$
$\therefore d = 20.6$ (3 s.f.)

E.g. 4 The board of examiners have decided that $80\,\%$ of all candidates sitting A level maths will obtain a pass grade. The actual exam marks are found to be normally distributed with a mean of 45 and a standard deviation of 7.

(a) What is the lowest score a student can get on the exam to be awarded a pass grade?

(b) Given that $10\,\%$ of students will achieve an A*, calculate the lowest mark required to get the highest grade.

Give your answers to the nearest mark.

Working: (a) $X \sim \text{N}(45,\, 7^2)$

Let p be the pass mark: $P(X > p) = 0.8$

$P(X < p) = 1 - 0.8 = 0.2 \quad \Rightarrow \quad p = 39.1$

The pass mark is 39.

(b) Let a be the mark required to get an A*: $P(X > a) = 0.1$

$P(X < a) = 1 - 0.1 = 0.9 \quad \Rightarrow \quad a = 54.0$

The mark required to achieve an A* is 54.

E.g. 5 Batteries for a radio have a mean life of 160 hours and a standard deviation of 30 hours. Assuming the battery life follows a normal distribution, calculate:

(a) the proportion of batteries which have a life between 150 and 180 hours

(b) the range, symmetrical about the mean, within which $75\,\%$ of batteries lie.

Working: (a) $X \sim \text{N}(160,\, 30^2)$

$P(150 < X < 180) = 0.378$

The proportion of batteries which have a life between 150 and 180 hours is 0.378.

(b) $P(160 - x < X < 160 + x) = 0.75$ ***symmetry about the mean***

$P(160 < X < 160 + x) = 0.375$

$\Rightarrow \quad P(X < 160 + x) = 0.875$ ***since*** $P(X < 160) = 0.5$

$160 + x = 194.5 \quad \Rightarrow \quad x = 34.5$

The range is 125.5 to 194.5 hours.

Video: Inverse normal function

Normal distribution EQ

Solutions to Starter and E.g.s

Exercise

p386 17C Qu 1i, 2-7, (8-11 red)