Limitations of the Newton-Raphson Method

Starter

1. (Review of last lesson) Let $f(x) = e^x - 10x$.
   (a) Show that there is a root between $x = 3$ and $x = 4$.
   (b) Solve the equation $f(x) = 0$ to 3 d.p.
   Hint: the derivative of e^x is e^x

Working: (a) $f(3) = -9.914\ldots < 0$ and $f(4) = 14.598\ldots > 0$
Since there is a sign change between $x = 3$ and $x = 4$
and the curve is continuous, there is a root between $x = 3$ and $x = 4$.

(b) $f(x) = e^x - 10x \quad \Rightarrow \quad f'(x) = e^x - 10$

Substitute into the formula: $x_{n+1} = x_n - \dfrac{e^{x_n} - 10x_n}{e^{x_n} - 10}$

N.B. Remember to replace x by x_n in the formula

Substitute $x_0 = 3.5$ into formula: $x_1 = 3.5 - \dfrac{e^{3.5} - 10 \times 3.5}{e^{3.5} - 10}$

$x_1 = 3.5815...$

N.B. Use the back button and replace 3.5 by ANS in the formula

Substitute x_1 into the formula: $x_2 = 3.5771...$

N.B. Now press the '=' button to get subsequent iterations

Substitute x_2 into the formula: $x_3 = 3.5771...$

Since $x_2 = x_3$, this is the value of the root we require.

So $x = 3.577$ (3 d.p.)

N.B. Your iterations may be different if you chose a different starting point to $x_0 = 3.5$.

2. (Review of last lesson) Use the Newton-Raphson method to find a root for $2x^3 - 15x^2 + 109 = 0$ to 5 s.f. starting with $x_0 = 1$.

Working: Let $f(x) = 2x^3 - 15x^2 + 109 \quad \Rightarrow \quad f'(x) = 6x^2 - 30x$

Substitute into the formula: $x_{n+1} = x_n - \dfrac{2{x_n}^3 - 15{x_n}^2 + 109}{6{x_n}^2 - 30x_n}$

N.B. Remember to replace x by x_n in the formula

Substitute $x_0 = 1$ into formula: $x_1 = 1 - \dfrac{2 \times 1^3 - 15 \times 1^2 + 109}{6 \times 1^2 - 30 \times 1}$

$x_1 = 5$

N.B. Use the back button and replace 1 by ANS in the formula

Substitute x_1 into the formula: $x_2 =$ Math ERROR

Something has gone wrong — see below.

E.g. 1 Why does the Newton-Raphson method not work when the starting value is a stationary value? Explain with reference to:
(a) the formula (b) a graph.

Working: (a) At a stationary point $f'(x_n) = 0$ so the denominator of the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ is zero and we cannot divide by zero.

(b) At a stationary point, the tangent to the curve is horizontal so will not meet the $x-$axis.

E.g. 2 Let $g(x) = x^3 - 4x + 1$. Explain why the Newton-Raphson method fails when $x_0 = \dfrac{2\sqrt{3}}{3}$.

Working: $g'(x) = 3x^2 - 4$
Solving $g'(x) = 0$ gives $3x^2 - 4 = 0$
$x^2 = \dfrac{4}{3}$ so $x = \pm\dfrac{2\sqrt{3}}{3}$
$x = \dfrac{2\sqrt{3}}{3}$ is a stationary point of the function $g(x) = x^3 - 4x + 1$
Therefore, if this is the x_0 value the tangent to the curve would be horizontal and so not intersect the $x-$axis i.e. the Newton-Raphson method fails

E.g. 3 (a) Show that the equation $x^4 - 4x^3 - 7.5x^2 + 50x - 55 = 0$ has a root between 2 and 3.

(b) Explain why $x_0 = 2.5$ is not a suitable starting point for a Newton-Raphson iteration to find this root.

(c) Use the starting value of $x_0 = 2.6$ to find the root correct to 3 d.p.

Working: (a) Let $f(x) = x^4 - 4x^3 - 7.5x^2 + 50x - 55$

$f(2) = -1 < 0$

$f(3) = 0.5 > 0$

Since there is a sign change, there is a root between 2 and 3.

(b) $f'(x) = 4x^3 - 12x^2 - 15x + 50$

$f'(2.5) = 0$ so the Newton-Raphson method fails because 2.5 is a stationary point and so the tangent to the curve would be horizontal and so not intersect the $x-$axis

(c) ***Formula:*** $x_{n+1} = x_n - \dfrac{{x_n}^4 - 4{x_n}^3 - 7.5{x_n}^2 + 50x_n - 55}{4{x_n}^3 - 12{x_n}^2 - 15x_n + 50}$

N.B. Remember to replace x by x_n in the formula

$x_0 = 2.6$:

$$x_1 = 2.6 - \frac{2.6^4 - 4 \times 2.6^3 - 7.5 \times 2.6^2 + 50 \times 2.6 - 55}{4 \times 2.6^3 - 12 \times 2.6^2 - 15 \times 2.6 + 50}$$

$x_1 = 4.2652$ (4 s.f.)

N.B. Use the back button and replace 2.6 by ANS in the formula

Substitute x_1 into the formula: $x_2 = 3.7223$ (4 s.f.)

N.B. Now press the '$=$' button to get subsequent iterations

Substitute x_2 into the formula: $x_3 = 3.3457$ (4 s.f.)

Substitute x_3 into the formula: $x_4 = 3.0954$ (4 s.f.)

Substitute x_4 into the formula: $x_5 = 2.9460$

Substitute x_5 into the formula: $x_6 = 2.8801$ (4 s.f.)

Substitute x_6 into the formula: $x_7 = 2.8667$ (4 s.f.)

Substitute x_7 into the formula: $x_8 = 2.8661$ (4 s.f.)

Substitute x_8 into the formula: $x_9 = 2.8661$ (4 s.f.)

Since $x_8 = x_9$, this is the value of the root we require.

So $x = 2.866$ (4 s.f.)

Video: Limitations of Newton-Raphson

Video: Newton-Raphson method

Solutions to Starter and E.g.s

Exercise

p310 14C Qu 2, 4, 6

p327 MP14 Qu 4, 6