Modelling with Differential Equations

Starter

1. **(Review of last lesson)**

Solve the equation $\frac{dy}{dx} = 2xy + 5x$ given that $y = -1$ when $x = 0$.

Working: $\frac{dy}{dx} = 2xy + 5x \quad \Rightarrow \quad \frac{dy}{dx} = x(2y+5)$

Separate the variables: $\int \frac{1}{2y+5} dy = \int x dx$

Integrate both sides: $\frac{1}{2} \ln|2y+5| = \frac{1}{2}x^2 + c$

Multiply by 2: $\ln|2y+5| = x^2 + A$

When $x = 0, y = -1$: $\ln 3 = A$

So $\ln|2y+5| = x^2 + \ln 3 \quad \Rightarrow \quad \ln|2y+5| - \ln 3 = x^2$

2nd law of logs: $\ln\left|\frac{2y+5}{3}\right| = x^2$

Convert to index form: $\frac{2y+5}{3} = e^{x^2}$

Rearrange: $y = \frac{3e^{x^2} - 5}{2}$

2. Show the following as differential equations:

(a) The number of bacteria, b, in a petri dish is increasing over time, t, at a rate directly proportional to the number of bacteria.

(b) The volume of a jelly, V, is decreasing over time, t, at a rate that is inversely proportional to the square of its volume.

Working: (a) $\frac{db}{dt} = kb$ where $k > 0$

(b) $\frac{dV}{dt} = -\frac{k}{V^2}$ where $k > 0$

E.g. 1 A colony of ants grows at a rate proportional to the size of the population, N.

(a) Express this as a differential equations relating N, t and k where t is the time in days since the colony was formed and k is a constant.

(b) When $N = 200$, the rate of increase of the population is 75. Find the value of k.

(c) The colony has an initial population of 200. How many ants will there be after 10 days?

Working: (a) $\frac{dN}{dt} \propto N \quad \Rightarrow \quad \frac{dN}{dt} = kN$

(b) When $n = 200$, $\frac{dN}{dt} = 75$

So $200 = 75k$ and $k = 0.375$

(c) $\frac{dN}{dt} = 0.375N \quad \Rightarrow \quad \int \frac{1}{N} dN = \int 0.375 dt$

$\ln N = 0.375t + c$

$N = e^{0.375t+c} = Ae^{0.375t}$

When $t = 0, N = 200$: $\quad A = 200$

$\therefore N = 200e^{0.375t}$;

When $t = 10$, $\therefore N = 200e^{0.375 \times 10} = 8504$ ants.

Identifying limitations

- Is there any information missing from the model?
- What happens when time gets very big? Does the model tend towards a fixed value? Is this value appropriate?
- Is the model appropriate? E.g. Is it continuous when the variable is discrete? Does it allow negative values that don't make sense?
- Other factors: seasonal variation, food limits, natural immunity from disease, immigration/emigration

E.g. 2 At time t minutes the rate of change of temperature of an object as it cools is proportional to the temperature T°C of the object at the time.

(a) Given that $T = 60$°C when $t = 0$, show that $T = 60e^{-kt}$, where k is a positive constant.

(b) Given also that $T = 40$°C, when $t = 5$ minutes, find the temperature of the object after 20 minutes.

(c) Explain why your answer to the final part may be incorrect in the real world.

Working: (a) $\frac{dT}{dt} \propto -T \quad \Rightarrow \quad \frac{dT}{dt} = -kT$

Separate the variables: $\int \frac{1}{T} dt = -\int k dt$

Integrate both sides: $\ln T = -kt + c$

$T = e^{-kt+c} = Ae^{-kt}$

When $t = 0, T = 60$: $60 = Ae^0 \quad \Rightarrow \quad A = 60$

(b) When $t = 5, T = 40$: $40 = 60e^{-5k}$

$\frac{2}{3} = e^{-5k}$

$\ln \frac{2}{3} = -5k$

$\therefore k = -\frac{1}{5} \ln \frac{2}{3} = \frac{1}{5} \ln \frac{3}{2} \approx 0.0811$

When $t = 20$: $T = 60e^{-0.0811 \times 20} = 11.9$°C (3 s.f.)

(c) Room temperature is usually about 20°C so the temperature of the object would not go below this value.

Video: Differential equations (direct proportion)
Video: Differential equations (inverse proportion)

Differential equations EQ

Solutions to Starter and E.g.s

Exercise
p292 13C Qu 1i, 2-11

Summary

Rate of ***increase*** of a quantity x is proportional to the amount of x: $\frac{dx}{dt} = kx$

Rate of ***decrease*** of a quantity x is proportional to the amount of x: $\frac{dx}{dt} = -kx$

where $k > 0$.

Limitations:

What happens when $t \to \infty$?
Think disease, food sources, predators etc.