Modelling with trigonometric functions

Starter

1. Give the equation of the curve $y = \cos x$ after it has undergone these transformations:
 (a) Horizontal stretch, factor 3, followed by a vertical translation of 7 units down
 (c) Vertical translation of 9 units down, followed by a vertical stretch, factor 2
 (d) Horizontal stretch, factor $\frac{1}{4}$ followed by a horizontal translation 6 units left

Working: (a) $y = \sin\left(\frac{1}{3}x\right) - 7$

(c) Is it $y = 2\sin x - 9$ or $y = 2(\sin x - 9)$?
Vertical transformations follow ***BIDMAS***
$y = 2\sin x - 9 \Rightarrow$ multiply then add $\Rightarrow$ stretch then translation
$y = 2(\sin x - 9) \Rightarrow$ bracket then multiply $\Rightarrow$ translation then stretch
Answer is $y = 2(\sin x - 9)$

(d) Is it $y = \sin(4x + 6)$ or $y = \sin 4(x + 6)$?
Horizontal transformations follow ***reverse BIDMAS***
$y = \sin(4x + 6) \Rightarrow$ add then multiply $\Rightarrow$ translation then stretch
$y = \sin 4(x + 6) \Rightarrow$ multiply then bracket $\Rightarrow$ stretch then translate
Answer is $y = \sin(4(x + 6))$

2. The graph of $y = \sin x$ has:
 Centre line at $y = 0$ (i.e. the $x-$axis)
 Amplitude of 1 (distance from centre line to a maximum or minimum)
 Period of $360°$ (length of 1 complete wave)
 Maximum value of 1 and minimum value of -1

For these curves state
(i) the equation of the centre line,
(ii) the amplitude,
(iii) the period,
(iv) the maximum and minimum values.
(a) $y = 3\sin 6x + 8$ (b) $y = a\sin bx + d$

Hint: write down the transformations.

Working: (a) $y = 3\sin 6x + 8$
Vertical stretch, factor 3; then vertical translation, 8 units up.
Horizontal stretch, factor $\frac{1}{6}$
(i) The equation of the centre line is $y = 8$.
(ii) The amplitude is 3.
(iii) The period is $\frac{360}{6} = 60°$.
(iv) The maximum value is 11 and the minimum value is 5.

(b) $y = a \sin bx + d$

Vertical stretch, factor a; then vertical translation, d units up.

Horizontal translation, c units left; then horizontal stretch, factor $\frac{1}{b}$

(i) The equation of the centre line is $y = d$.

(ii) The amplitude is a.

(iii) The period is $\frac{360}{b}$.

(iv) The maximum value is $d + a$ and the minimum value is $d - a$.

E.g. 1 The height of a tide can be modelled by a function of the form $h = a\cos bt° + c$, where h is the height in metres of the water and t is the time in hours after midnight. Find the values of a, b and c.

Tide	Time	Height (m)
High	00:00	14
Low	12:00	6

Working: Centre line is half-way between high and low points: $c = \frac{14+6}{2} = 10$

Amplitude is distance from centre line to maximum: $a = 4$

The period is 24: $24 = \frac{360}{b}$ $\Rightarrow$ $b = 15$

$\therefore h = 4\cos 15t° + 10$

E.g. 2 The water levels in a dockyard follow a $12-$hour cycle, and can be modelled by the equation $D = A + B\sin 30t°$, where D is the depth of the water in metres, A and B are positive constants, and t is the time in hours after the start of the working day at 8am. Given that the greatest and least depths of water in the dock are 7.8 m and 2.2 m, find:

(a) the value of A and the value of B

(b) the depth of water, to the nearest cm, in the dock at noon.

Working: (a) A is the centre line $\Rightarrow$ $A = \frac{7.8+2.2}{2} = 5$

B is the amplitude $\Rightarrow$ $B = 7.8 - 5 = 2.8$

(b) $D = 5 + 2.8\sin 30t°$

At noon, $t = 4$: $D = 5 + 2.8\sin(30 \times 4) \approx 7.42$ cm

The depth of water in the dock at noon is 7.42 m (nearest cm)

E.g. 3 The depth of water in a harbour, y metres, can be modelled by the equation $y = 5\sin(30t)^{\circ} + 12$, where t is the time in hours from midnight on a particular day.

(a) Sketch the graph of this function over a period of 24 hours.

(b) Using your graph, or otherwise, state the times of high and low tides.

(c) Find the length of time for which the depth of water in the harbour is greater than 15 metres and the times during the day between which this occurs.

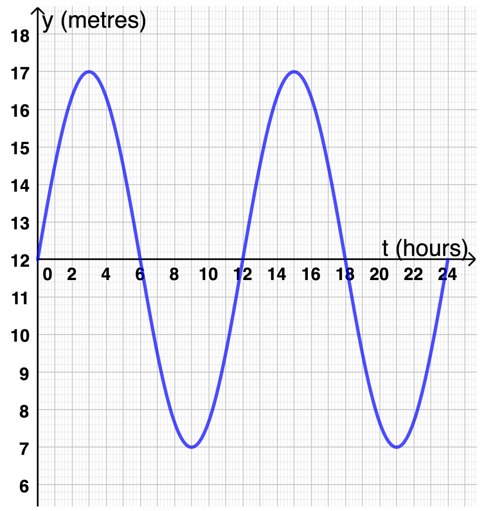
Working: (a) Centre line is at $y = 12$

Amplitude is 5

Maximum at $12 + 5 = 17$

Minimum at $12 - 5 = 7$

Period $= \dfrac{360}{30} = 12$ so two full waves over a 24 hour period



(b) From the graph, high tide is at $03:00$ and $15:00$ and low tide is at $09:00$ and $21:00$.

(c) From the graph there are two periods of time when the depth is greater than 15 m.

Solve $5\sin(30t)^{\circ} + 12 = 15 \quad \Rightarrow \quad \sin(30t)^{\circ} = \dfrac{3}{5}$

$\sin^{-1}\left|\dfrac{3}{5}\right| \approx 36.9^{\circ}$

$\dfrac{3}{5}$ is +ve $\quad \Rightarrow \quad$ **A** and **S** quadrants

$30t = 36.9 \quad \Rightarrow \quad t \approx 1.229 = 01:14$ (nearest minute)

$30t = 180 - 36.9 \quad \Rightarrow \quad t \approx 4.771 = 04:46$ (nearest minute)

Add 12 hours to find the second times during the day

Total time $= 2 \times (4.771 - 1.229) = 7$h 5m

Times: between $01:14$ and $04:46$

between $13:14$ and $16:46$

E.g. 4 At a certain latitude, the number d of hours of daylight each day of the year is taken to be $d = A + B\sin kt°$, where A, B and k are constants and t is the time in days after the spring equinox.

(a) Assuming the number of hours of daylight follows an annual cycle of 365 days, find the exact value of k.

(b) Given that the longest and shortest days have 6 and 18 hours of daylight respectively, state the values of A and B.

(c) Find, in hours and minutes, the amount of daylight on New Year's Day, which is 80 days before the spring equinox.

(d) A town at this latitude holds a fair twice a year on those days having 10 hours of daylight. Find, in relation to the spring equinox, which two days these are.

Working: (a) Period is 365: $365 = \frac{360}{k} \Rightarrow k = \frac{360}{365} = \frac{72}{73}$

(b) Centre line is at $y = A$: $A = \frac{18+6}{2} = 12$

B is the amplitude: $B = \frac{18-6}{2} = 6$

(c) $d = 12 + 6\sin\frac{72}{73}t°$

80 days before the spring equinox $\Rightarrow t = 365 - 80 = 285$

$d = 12 + 6\sin\frac{72}{73} \times 285 \approx 6.11\text{h} = 6\text{h } 7\text{m}$

(d) $d = 10$: $12 + 6\sin\frac{72}{73}t° = 10$

$\sin\frac{72}{73}t° = -\frac{1}{3}$

$\sin^{-1}\left|-\frac{1}{3}\right| \approx 19.47°$

$-\frac{1}{3}$ is $-$ve $\Rightarrow$ **T** and **C** quadrants

$\frac{72}{73}t = 180 + 19.47 \Rightarrow t \approx 202.2 \Rightarrow$ Day 203

$\frac{72}{73}t = 180 - 19.47 \Rightarrow t \approx 345.3 \Rightarrow$ Day 346

Video: Modelling with trigonometric functions A
Video: Modelling with trigonometric functions B

Trigonometric transformations EQ

Solutions to Starter and E.g.s

Exercise
p144 7C Qu 1, 2i, 3-11, (12 red)