Parametric Equations

Starter

1. **(Review of last lesson)** Decide whether the curve $y = x^2 - \frac{1}{x}$ has a point of inflexion.

Working: $y = x^2 - \frac{1}{x} = x^2 - x^{-1}$

$\frac{dy}{dx} = 2x + x^{-2}$

$\frac{d^2y}{dx^2} = 2 - 2x^{-3}$

A possible PoI occurs when $\frac{d^2y}{dx^2} = 0$: $\quad 2 - 2x^{-3} = 0$

$2 = \frac{2}{x^3} \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad x = 1$ is a ***possible*** PoI

Now find the gradient change either side of the possible PoI.

When $x = 0.9$, $\quad \frac{dy}{dx} = 2 \times 0.9 + \frac{1}{0.9^2} > 0$

When $x = 1.1$, $\quad \frac{dy}{dx} = 2 \times 1.1 + \frac{1}{1.1^2} > 0$

Since the gradient is the same sign, $x = 1$ is a point of inflexion

2. **(Review of last lesson)**

If $f(x) = \frac{1}{16}x^4 + \frac{3}{4}x^3 - \frac{21}{8}x^2 - 6x + 20$, identify the range of values of x for which the graph of $y = f(x)$ is concave and convex. Express your answers in set notation.

Working: $f'(x) = \frac{1}{4}x^3 + \frac{9}{4}x^2 - \frac{21}{4}x - 6$

$f''(x) = \frac{3}{4}x^2 + \frac{9}{2}x - \frac{21}{4}$

A curve is convex when $f''(x) > 0$: $\quad \frac{3}{4}x^2 + \frac{9}{2}x - \frac{21}{4} > 0$

Multiply by 4, divide by 3: $\quad x^2 + 6x - 7 > 0$

Solving $x^2 + 6x - 7 = 0 \quad (x+7)(x-1) = 0$

$x = -7$ and $x = 1$

The curve $x^2 + 6x - 7 = 0$ is concave-up

$> 0 \quad \Rightarrow \quad$ above the $x-$axis

The curve is convex when $\{x : x < -7\} \cup \{x : x > 1\}$.

It follows that the curve is concave when $\{x : x > -7\} \cap \{x : x < 1\}$.

3. Find an expression involving y and x but no t given that:

(a) $x = 3t$ and $y = \dfrac{6}{t}$

(b) $x = 4t^2$ and $y = 8t$

Working: (a) Substitute $t = \dfrac{x}{3}$ into $y = \dfrac{6}{t}$: $y = \dfrac{6}{\left(\dfrac{x}{3}\right)} \Rightarrow y = \dfrac{18}{x}$

(b) Substitute $t = \dfrac{y}{8}$ into $x = 4t^2$: $x = 4\left(\dfrac{y}{8}\right)^2 \Rightarrow y^2 = 16x$

E.g. 1 Express $x = 3\cos t$, $y = 4\sin t$ in cartesian form.

Working:

$\cos^2 t + \sin^2 t = 1$

so $\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 1$

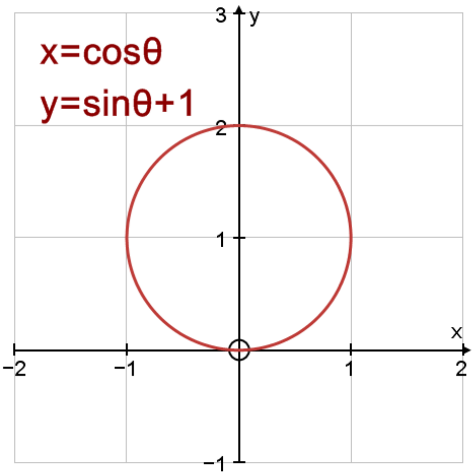
$\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

$16x^2 + 9y^2 = 144$

E.g. 2 Sketch the graph given by $x = \cos\theta$ and $y = \sin\theta + 1$.

ϑ	0	π/2	π	3π/2	2π
x	1	0	−1	0	1
y	1	2	1	0	1

Working: Circle, centre (0, 1) and radius 1



E.g. 3 Transform these parametric curves into cartesian form:

(a) $x = \dfrac{5}{2-t}, \quad y = \dfrac{4}{2-t}$

(b) $x = \cos t, \quad y = 4\sin 2t \sin t$

Working: (a) From $x = \dfrac{5}{2-t}$, $2 - t = \dfrac{5}{x}$

Similarly $2 - t = \dfrac{4}{y}$

Equating gives $\dfrac{5}{x} = \dfrac{4}{y} \quad \Rightarrow \quad 5y = 4x$

(b) $y = 4\sin 2t \sin t$

$= 4 \times 2\sin t \cos t \times \sin t$ ***since*** $\sin 2t = 2\sin t \cos t$

$= 8\cos t \sin^2 t$

$= 8\cos t(1 - \cos^2 t)$ ***since*** $\sin^2 t + \cos^2 t = 1$

$y = 8x(1 - x^2)$ ***since*** $x = \cos t$

E.g. 4 The curve C is defined by the parametric equations $x = 2t^2 - 7t, \; y = 10 - t^2$.

(a) Find the value of a if $(a, \; 1)$ is a point on the curve and $a > 1$.

(b) Decide whether $(-6, 4)$ lies on curve C.

Working: (a) Substitute $y = 1$ into $y = 10 - t^2$: $1 = 10 - t^2$

$\therefore t = \pm 3$

When $t = 3, x = 2 \times 3^2 - 7 \times 3 < 1$

When $t = -3, x = 2 \times (-3)^2 - 7 \times (-3) = 39 > 1$

So $a = 39$

(b) Substitute $x = -6$ into $x = 2t^2 - 7t$: $-6 = 2t^2 - 7t$

Solving $2t^2 - 7t + 6 = 0$: $(2t - 3)(t - 2) = 0$

when $x = -6, t = 1.5$ or 2

Substitute $t = 1.5$ into $y = 10 - t^2$ gives $y = 7.75 \neq 4$

Substitute $t = 2$ into $y = 10 - t^2$ gives $y = 6 \neq 4$

So the point $(-6, 4)$ does not lie on the curve C

Video: Introduction to parametric functions

Parametric functions EQ

Solutions to Starter and E.g.s

Exercise

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