Points of Inflexion

Starter

1. **(Review of last lesson)** Evaluate $\int_1^2 \frac{3x+5}{x(x+10)} dx$ giving your answer to 3 s.f.

Working: $\frac{3x+5}{x(x+10)} \equiv \frac{A}{x} + \frac{B}{x+10}$

$3x+5 = A(x+10) + Bx$

When $x = -10$: $\quad -25 = -10B \quad \Rightarrow \quad B = \frac{5}{2}$

When $x = 0$: $\quad 5 = 10A \quad \Rightarrow \quad A = \frac{5}{10} = \frac{1}{2}$

$$\int_1^2 \frac{3x+5}{x(x+10)} dx = \int_1^2 \frac{1}{2x} + \frac{5}{2(x+10)} dx$$
$$= \left[\frac{1}{2}\ln|x| + \frac{5}{2}\ln|x+10|\right]_1^2$$
$$= \left(\frac{1}{2}\ln 2 + \frac{5}{2}\ln 12\right) - \left(\frac{1}{2}\ln 1 + \frac{5}{2}\ln 11\right)$$
$$= \frac{1}{2}\ln 2 + \frac{5}{2}\ln\frac{12}{11}$$
$$= 0.564$$

2. **(Review of AS material)** Consider the curve $y = x^3$. Find

(a) the stationary point

(b) the value of the second derivative at the stationary point

(c) the signs of the gradient to the left and right of the stationary point.

Working: (a) $\frac{dy}{dx} = 3x^2$

When $\frac{dy}{dx} = 0,\ 3x^2 = 0 \quad \Rightarrow \quad x = 0$

When $x = 0,\ y = 0$ so the stationary point is at $(0, 0)$

(b) $\frac{d^2y}{dx^2} = 6x$

When $x = 0,\ \frac{d^2y}{dx^2} = 0$

(c) When $x = -0.1,\ \frac{dy}{dx} = 3 \times (-0.1)^2 > 0$

When $x = 0.1,\ \frac{dy}{dx} = 3 \times 0.1^2 > 0$

The signs of the gradient either side of the stationary point are positive i.e. the gradient change is +ve/0/+ve

3. **(Review of AS material)** Repeat question 2 for the curve $y = x^4$.

Working: (a) $\dfrac{dy}{dx} = 4x^3$

When $\dfrac{dy}{dx} = 0, 4x^3 = 0 \quad \Rightarrow \quad x = 0$

When $x = 0, y = 0$ so the stationary point is at $(0, 0)$

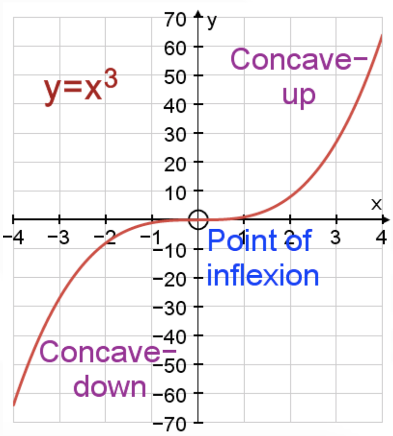
(b) $\dfrac{d^2y}{dx^2} = 12x^2$

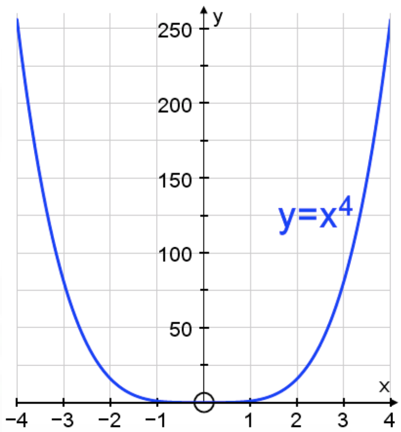
When $x = 0, \dfrac{d^2y}{dx^2} = 0$

(c) When $x = -0.1, \dfrac{dy}{dx} = 4 \times (-0.1)^3 < 0$

When $x = 0.1, \dfrac{dy}{dx} = 4 \times 0.1^3 > 0$

The sign to the left of the stationary point is negative but to the right it is positive i.e. the gradient change is —ve/0/+ve





E.g. 1 Find and classify the stationary point(s) and point(s) of inflexion of the curve $f(x) = x^4 - 4x^3$. Hence sketch the graph.

Working:

$f'(x) = 4x^3 - 12x^2$

$f'(x) = 0$ when $x = 3$ and $x = 0$ ***these are stationary points***

$f''(x) = 12x^2 - 24x$

$f''(3) > 0$ so when $x = 3$ there is a minimum

When $x = 3, y = -27$ so (3, —27) is a minimum

$f''(0) = 0$ so $x = 0$ is a ***possible*** point of inflexion

Solving $f''(x) = 0$ gives $x = 0$ and $x = 2$ — possible points of inflexion

Now choose $x-$values either side of $x = 0$ and $x = 2$ to check gradient

$x = 0$: $\quad f'(-0.1) < 0$

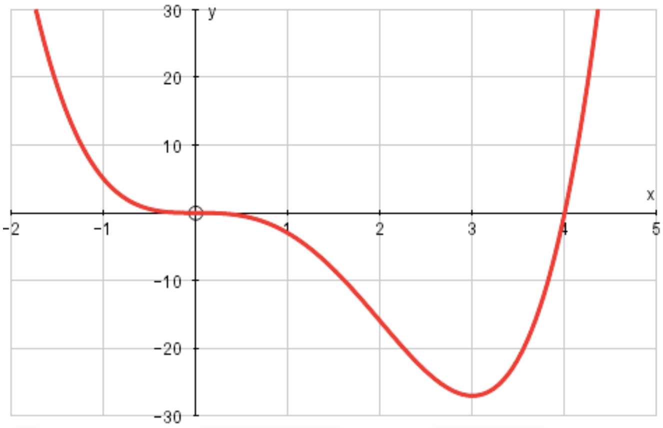
$\quad f'(0.1) < 0$

When $x = 0, y = 0$ so $(0, 0)$ is a point of inflection

$x = 2$: $\quad f'(1.9) < 0$

$\quad f'(2.1) < 0$

When $x = 2, y = -16$ so $(2, -16)$ is a point of inflection



E.g. 2 Find the point(s) of inflexion of the graph of $y = 2x^4 + 4x^3 - 72x^2$.

Working: $\dfrac{dy}{dx} = 8x^3 + 12x^2 - 144x$

$\dfrac{d^2y}{dx^2} = 24x^2 + 24x - 144$

A possible PoI occurs when $\dfrac{d^2y}{dx^2} = 0$, $24x^2 + 24x - 144 = 0$

$x^2 + x - 6 = 0 \quad \Rightarrow \quad (x-2)(x+3) = 0$

$\therefore x = -3$ and $x = 2$ are ***possible*** PoI

Now find the gradient change either side of the possible PoI.

When $x = -3.1$, $\dfrac{dy}{dx} = 8 \times (-3.1)^3 + 12 \times (-3.1)^2 - 144 \times (-3.1) > 0$

When $x = -2.9$, $\dfrac{dy}{dx} = 8 \times (-2.9)^3 + 12 \times (-2.9)^2 - 144 \times (-2.9) > 0$

Since the gradient is the same sign, $x = -3$ is a point of inflexion

When $x = 1.9$, $\dfrac{dy}{dx} = 8 \times 1.9^3 + 12 \times 1.9^2 - 144 \times 1.9 < 0$

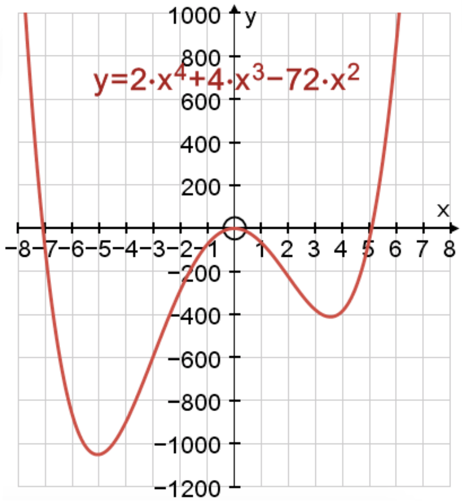
When $x = 2.1$, $\dfrac{dy}{dx} = 8 \times 2.1^3 + 12 \times 2.1^2 - 144 \times 2.1 < 0$

Since the gradient is the same sign, $x = 2$ is a point of inflexion

When $x = -3$, $y = 2(-3)^4 + 4(-3)^3 - 72(-3)^2 = -594$

When $x = 2$, $y = 2 \times 2^4 + 4 \times 2^3 - 72 \times 2^2 = -224$

The point of inflection are $(-3, -594)$ and $(2, -224)$



E.g. 3 Find the range of $x-$values for which the curve $y = x^3 + x^2 - x$ is convex.

Working: $\dfrac{dy}{dx} = 3x^2 + 2x - 1$

$\dfrac{d^2y}{dx^2} = 6x + 2$

The curve is convex when $\dfrac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad 6x + 2 > 0$

$x > -\dfrac{1}{3}$

E.g. 4 Find the values for which the curve $f(x) = \dfrac{1}{3}x^3 - x^2 - 15x$ is concave:

Working: $\dfrac{dy}{dx} = x^2 - 2x - 15$

$\dfrac{d^2y}{dx^2} = 2x - 2$

The curve is concave when $\dfrac{d^2y}{dx^2} < 0 \quad \Rightarrow \quad 2x - 2 < 0$

$x < 1$

Video: Points of inflexion
Video: Convex and concave curves

Solutions to Starter and E.g.s

Exercise
p252 12A Qu 4-10