

Points of Inflexion

Starter

1. **(Review of last lesson)** Evaluate $\int_1^2 \frac{3x+5}{x(x+10)} dx$ giving your answer to 3 s.f.

Working:
$$\frac{3x+5}{x(x+10)} \equiv \frac{A}{x} + \frac{B}{x+10}$$
$$3x+5 = A(x+10) + Bx$$

When $x = -10$: $-25 = -10B \Rightarrow B = \frac{5}{2}$

When $x = 0$: $5 = 10A \Rightarrow A = \frac{5}{10} = \frac{1}{2}$

$$\begin{aligned} \int_1^2 \frac{3x+5}{x(x+10)} dx &= \int_1^2 \left(\frac{1}{2x} + \frac{5}{2(x+10)} \right) dx \\ &= \left[\frac{1}{2} \ln|x| + \frac{5}{2} \ln|x+10| \right]_1^2 \\ &= \left(\frac{1}{2} \ln 2 + \frac{5}{2} \ln 12 \right) - \left(\frac{1}{2} \ln 1 + \frac{5}{2} \ln 11 \right) \\ &= \frac{1}{2} \ln 2 + \frac{5}{2} \ln \frac{12}{11} \\ &= 0.564 \end{aligned}$$

2. **(Review of AS material)** Consider the curve $y = x^3$. Find
- the stationary point
 - the value of the second derivative at the stationary point
 - the sign of the gradient to the left and right of the stationary point.

Working: (a) $\frac{dy}{dx} = 3x^2$

When $\frac{dy}{dx} = 0$, $3x^2 = 0 \Rightarrow x = 0$

When $x = 0$, $y = 0$ so the stationary point is at $(0, 0)$

(b) $\frac{d^2y}{dx^2} = 6x$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

(c) When $x = -0.1$, $\frac{dy}{dx} = 3 \times (-0.1)^2 > 0$

When $x = 0.1$, $\frac{dy}{dx} = 3 \times 0.1^2 > 0$

The signs of the gradient either side of the stationary point are positive i.e. the gradient change is +ve/0/+ve

3. (Review of AS material) Repeat question 2 for the curve $y = x^4$.

Working: (a) $\frac{dy}{dx} = 4x^3$

When $\frac{dy}{dx} = 0$, $4x^3 = 0 \Rightarrow x = 0$

When $x = 0$, $y = 0$ so the stationary point is at $(0, 0)$

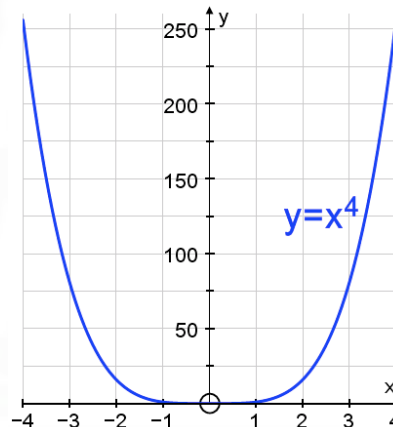
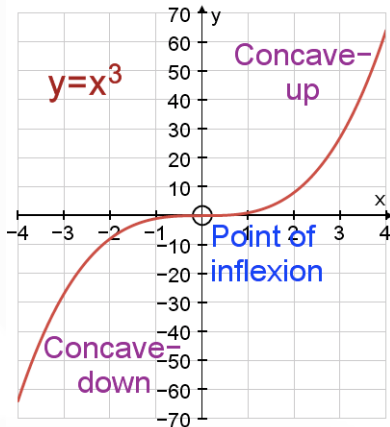
(b) $\frac{d^2y}{dx^2} = 12x^2$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

(c) When $x = -0.1$, $\frac{dy}{dx} = 4 \times (-0.1)^3 < 0$

When $x = 0.1$, $\frac{dy}{dx} = 4 \times 0.1^3 > 0$

The sign to the left of the stationary point is negative but to the right it is positive i.e. the gradient change is $-ve/0/+ve$



E.g. 1 Find the range of x -values for which the curve $y = x^3 + x^2 - x$ is convex. Give your answer in set notation.

Working: $\frac{dy}{dx} = 3x^2 + 2x - 1$
 $\frac{d^2y}{dx^2} = 6x + 2$

The curve is convex when $\frac{d^2y}{dx^2} > 0 \Rightarrow 6x + 2 > 0$

$x > -\frac{1}{3}$

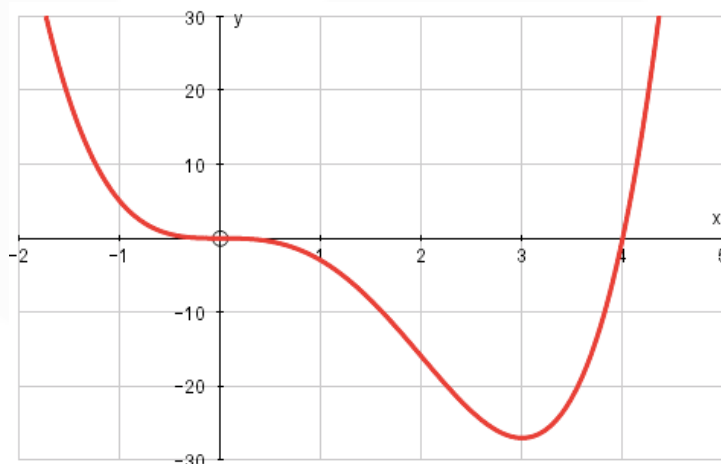
In set notation: $\left\{ x : x > -\frac{1}{3} \right\}$

E.g. 2 Find the values for which the curve $f(x) = x^4 + 4x^3 - 18x^2 + 7$ is concave. Give your answer in set notation.

Working: $f'(x) = 4x^3 + 12x^2 - 36x$
 $f''(x) = 12x^2 + 24x - 36 = 12(x^2 + 2x - 3)$
 The curve is concave when $f''(x) < 0 \Rightarrow 12(x^2 + 2x - 3) < 0$
Solving the quadratic inequality: $(x - 1)(x + 3) < 0$
 Roots are $x = 1$ and $x = -3$
 $< 0 \Rightarrow$ below the x -axis so $-3 < x < 1$
 In set notation: $\{x : x > -3\} \cap \{x : x < 1\}$

E.g. 3 Find and classify the stationary point(s) and point(s) of inflexion of the curve $f(x) = x^4 - 4x^3$. Hence sketch the graph.

Working: $f'(x) = 4x^3 - 12x^2$
 $f'(x) = 0$ when $x = 3$ and $x = 0$ **these are stationary points**
 $f''(x) = 12x^2 - 24x$
 $f''(3) > 0$ so when $x = 3$ there is a minimum
 When $x = 3, y = -27$ so $(3, -27)$ is a minimum
 $f''(0) = 0$ so $x = 0$ is a **possible** point of inflexion
 Solving $f''(x) = 0$ gives $x = 0$ and $x = 2$ — **possible** points of inflexion
 Now choose x -values either side of $x = 0$ and $x = 2$ and check the signs of the **second derivative**.
 $x = 0:$ $f''(-0.1) = 12 \times (-0.1)^2 - 24 \times (-0.1) = 2.52 > 0$
 $f''(0.1) = 12 \times 0.1^2 - 24 \times 0.1 = -2.28 < 0$
 Since there is sign change, $x = 0$ is point of inflexion.
 When $x = 0, y = 0$ so $(0, 0)$ is a point of inflexion
 $x = 2:$ $f''(1.9) = 12 \times 1.9^2 - 24 \times 1.9 = -2.28 < 0$
 $f''(2.1) = 12 \times 2.1^2 - 24 \times 2.1 = 2.52 > 0$
 Since there is sign change, $x = 2$ is point of inflexion.
 When $x = 2, y = -16$ so $(2, -16)$ is a point of inflection.



E.g. 4 Find the point(s) of inflexion of the graph of $y = 2x^4 + 4x^3 - 72x^2$.

Working:

$$\frac{dy}{dx} = 8x^3 + 12x^2 - 144x$$

$$\frac{d^2y}{dx^2} = 24x^2 + 24x - 144$$

A possible Pol occurs when $\frac{d^2y}{dx^2} = 0$, $24x^2 + 24x - 144 = 0$

$$x^2 + x - 6 = 0 \quad \Rightarrow \quad (x - 2)(x + 3) = 0$$

$\therefore x = -3$ and $x = 2$ are **possible** Pol

Now choose x -values either side of the **possible Pol** and check the signs of the **second derivative**.

When $x = -3.1$, $\frac{d^2y}{dx^2} = 24 \times (-3.1)^2 + 24 \times (-3.1) - 144 = 12.24 > 0$

When $x = -2.9$, $\frac{d^2y}{dx^2} = 24 \times (-2.9)^2 + 24 \times (-2.9) - 144 = -11.76 < 0$

Since the second derivative changes sign, $x = -3$ is a point of inflexion.

When $x = 1.9$, $\frac{d^2y}{dx^2} = 24 \times 1.9^2 + 24 \times 1.9 - 144 = -11.76 < 0$

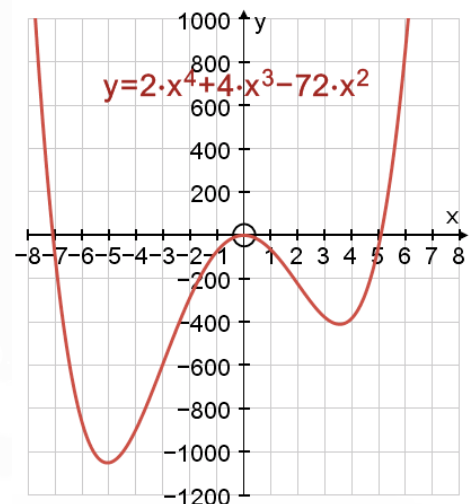
When $x = 2.1$, $\frac{d^2y}{dx^2} = 24 \times 2.1^2 + 24 \times 2.1 - 144 = 12.24 > 0$

Since the second derivative changes sign, $x = 2$ is a point of inflexion.

When $x = -3$, $y = 2(-3)^4 + 4(-3)^3 - 72(-3)^2 = -594$

When $x = 2$, $y = 2 \times 2^4 + 4 \times 2^3 - 72 \times 2^2 = -224$

The point of inflexion are $(-3, -594)$ and $(2, -224)$



Video: [Points of inflexion](#)

Video: [Convex and concave curves](#)

[Solutions to Starter and E.g.s](#)

Exercise

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