Proof by contradiction

Starter

1. *(Review of last lesson)* Prove that the square of an odd number is always odd.

Working: Let $2k+1$ be any odd number.

$$(2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

which is the form of an odd number.
So the square of an odd number is odd.

E.g. 1 Prove by contradiction that $\tan x - \sin x > 0$ for $0° < x < 90°$.

Working:

Opposite: Assume that $\tan x - \sin x \le 0$ for $0° < x < 90°$.

Working:
$\Rightarrow \quad \dfrac{\sin x}{\cos x} - \sin x \le 0$

$\Rightarrow \quad \sin x\left(\dfrac{1}{\cos x} - 1\right) \le 0$

$\Rightarrow$ For $0° < x < 90°$, $\sin x > 0$

$\Rightarrow \quad \dfrac{1}{\cos x} - 1 \le 0$

$\Rightarrow \quad \dfrac{1}{\cos x} \le 1$

Since for $0° < x < 90°$, $\cos x > 0$:

$\Rightarrow \quad \cos x \ge 1$

Contradiction: But this is a contradiction since for $0° < x < 90°$, $0 < \cos x < 1$.

Conclusion: Therefore, $\tan x - \sin x > 0$ for $0° < x < 90°$.

E.g. 2 Prove that there are an infinite number of even numbers.

Working:

Opposite: Assume that there are a finite number of even numbers.

Working:
$\Rightarrow$ There exists an N, which is the largest even number.

$\Rightarrow$ $N = 2a$ where a is an integer ($a \in \mathbb{Z}$)

$\Rightarrow$ $N + 2 = 2a + 2 = 2(a+1)$

Contradiction: But this is a contradiction since $a+1$ is an integer, so $N+2$ is an even number which is larger than N

Conclusion: Therefore, there are an infinite number of even numbers.

E.g. 3 Prove that there are an infinite number of prime numbers.

Working:

Opposite: Assume that there are a finite number of prime numbers.

Working: $\Rightarrow$ All the prime numbers can be written as $p_1, p_2, p_3, \ldots, p_n$ where p_n is the largest prime number.

Let P be the product of all the prime numbers.

$\Rightarrow \quad P = p_1 \times p_2 \times p_3 \times \ldots \times p_n$

$\Rightarrow \quad P + 1 = p_1 p_2 p_3 \ldots p_n + 1$

Divide $P + 1$ by p_1: $\dfrac{P+1}{p_1} = \dfrac{p_1 p_2 p_3 \ldots p_n + 1}{p_1}$

$\Rightarrow \quad \dfrac{P+1}{p_1} = \dfrac{p_1 p_2 p_3 \ldots p_n}{p_1} + \dfrac{1}{p_1}$

$\Rightarrow \quad \dfrac{P+1}{p_1} = p_2 p_3 \ldots p_n + \dfrac{1}{p_1}$

$\Rightarrow \quad P + 1$ gives a remainder of 1 when divided by p_1

$\Rightarrow \quad P + 1$ is not divisible by p_1

Similarly, $P + 1$ is not divisible by $p_2, p_3, \ldots, p_n$.

If $P + 1$ is prime, then it is a new prime not on the list.

If $P + 1$ is not prime, then it must be the product of primes not on the list.*

Contradiction: Hence, $p_1, p_2, p_3, \ldots, p_n$ is not a complete list of prime numbers.

Conclusion: Therefore, there are an infinite number of prime numbers.

*For example, imagine our list of primes is $2, 3, 5, 7, 11, 13.$
Then $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
i.e. $P + 1$ is the product of new primes not on the list

E.g. 4 Prove by contradiction that the product of a rational number with an irrational number is an irrational number.

Working: Let p be a rational number such that $p = \dfrac{a}{b}$, where a and $b \neq 0$ are integers and let q be an irrational number.

Assume the product of p and q is rational:

$\Rightarrow \quad pq = \dfrac{c}{d}$ where c and $d \neq 0$ are integers

$\Rightarrow \quad \dfrac{a}{b} \times q = \dfrac{c}{d} \qquad$ since $p = \dfrac{a}{b}$

$\Rightarrow \quad q = \dfrac{bc}{ad}$

But this is the form of a rational number, since bc and ad are integers, which is a contradiction since q is an irrational number.

Hence, the product of a rational number with an irrational number is an irrational number

E.g. 5 (a) Use proof by contradiction to prove that if p^2 is divisible by 2, then p is also divisible by 2.

(b) Write down a similar proof for 3.

(c) Does the proof work for the number 4? Explain your answer.

(d) If p^2 is divisible by k, then p is also divisible by k. Conjecture which numbers this is not always true for.

Working:

(a) Let p^2 be divisible by 2.
Assume p is not divisible by 2:
$\Rightarrow$ p is odd
$\Rightarrow$ p is of the form $2a + 1$ where a is an integer
$\Rightarrow$ $p^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$
But this is the form of an odd number and contradicts that p^2 is divisible by 2.
Therefore, by proof by contradiction, if p^2 is divisible by 2, then p is also divisible by 2.

(b) Let p^2 is divisible by 3.
Assume p is not divisible by 3:
$\Rightarrow$ p is of the form $3a + b$ where a is an integer and $b = 1$ or 2
$\Rightarrow$ $p^2 = (3a+b)^2 = 9a^2 + 6ab + b^2 = 3(3a^2 + 2ab) + b^2$
The first term is divisible by 3 but since $b^2 = 1$ or $b^2 = 4$ so b^2 is not divisible by 3.
Hence, the number $p^2 = 3(3a^2 + 2ab) + b^2$ is not divisible by 3.
But this contradicts that p^2 is divisible by 3.
Therefore, by proof by contradiction, if p^2 is divisible by 3, then p is also divisible by 3.

(c) Let p^2 is divisible by 4.
Assume p is not divisible by 4:
$\Rightarrow$ p is of the form $4a + b$ where a is an integer and $b = 1, 2$ or 3
$\Rightarrow$ $p^2 = (4a+b)^2 = 16a^2 + 8ab + b^2 = 4(4a^2 + 2ab) + b^2$
The first term is divisible by 4 and b^2 could be divisible by 4.
Therefore, the proof breaks down.
However, it is not a problem since we would not try to prove that $\sqrt{4}$ is irrational.

(d) It is not always true when k is a square number or when k is a multiple of a square number.

E.g. 6 Prove that $\sqrt{2}$ is an irrational number.

Working: **Opposite:** Assume that $\sqrt{2}$ is a rational number:

Working:
$\Rightarrow \quad \sqrt{2} = \frac{p}{q}$ where p and q are co-prime integers.
$\Rightarrow \quad 2 = \frac{p^2}{q^2}$
$\Rightarrow \quad p^2 = 2q^2$ so p^2 is a multiple of 2
$\Rightarrow \quad p$ is a multiple of 2, so $p = 2a$ where a is an integer
$\Rightarrow \quad (2a)^2 = 2q^2$
$\Rightarrow \quad 4a^2 = 2q^2$
$\Rightarrow \quad 2a^2 = q^2$ so q^2 is a multiple of 2
$\Rightarrow \quad q$ is a multiple of 2

Contradiction: But this is a contradiction because p and q are co-prime.

Conclusion: Therefore, $\sqrt{2}$ is an irrational number.

E.g. 7 Prove that $\sqrt{5}$ is irrational.

Working: **Opposite:** Assume that $\sqrt{5}$ is a rational number:

Working:
$\Rightarrow \quad \sqrt{5} = \frac{p}{q}$ where p and q are co-prime integers.
$\Rightarrow \quad 5 = \frac{p^2}{q^2}$
$\Rightarrow \quad p^2 = 5q^2$ so p^2 is a multiple of 5
$\Rightarrow \quad p$ is a multiple of 3, so $p = 3a$ where a is an integer
$\Rightarrow \quad (5a)^2 = 5q^2$
$\Rightarrow \quad 25a^2 = 5q^2$
$\Rightarrow \quad 5a^2 = q^2$ so q^2 is a multiple of 5
$\Rightarrow \quad q$ is a multiple of 5

Contradiction: But this is a contradiction because p and q are co-prime.

Conclusion: Therefore, $\sqrt{5}$ is an irrational number.

E.g. 8 Prove that $\log_3 5$ is irrational.

Working: **Opposite:** Assume that $\log_3 5$ is rational:

Working:
$\Rightarrow \quad \log_3 5 = \frac{p}{q}$ where p and q are co-prime integers.
$\Rightarrow \quad 5 = 3^{\frac{p}{q}}$
$\Rightarrow \quad 5^q = 3^p$

Contradiction: But this is a contradiction because the integer powers of 5 end in 5 and the integer powers of 3 end in $1, 3, 7$ or 9.

Conclusion: Therefore, $\log_3 5$ is irrational.

Video: Proof by contradiction
Solutions to Starter and E.g.s

Exercise
p4 1B Qu 1-9, (10-12 red)