Quotient Rule

Starter

1. **(Review of last lesson)** Differentiate $y = e^{-2x} \cos 5x$.

Working: Using the poetry
$$\frac{dy}{dx} = -2e^{-2x} \times \cos 5x - 5 \sin 5x \times e^{-2x}$$
$$\frac{dy}{dx} = -e^{-2x}(2\cos 5x + 5 \sin 5x)$$

2. We use the quotient rule to differentiate when we have one function divided by another i.e. $y = \dfrac{u}{v}$ where u and v are functions of x.

How could we change the function $y = \dfrac{u}{v}$ into a form where we could use the product rule?

Working: $y = \dfrac{u}{v} = u \times v^{-1}$

3. Hence using the product rule, and possibly the chain rule, find $\dfrac{dy}{dx}$ when $y = \dfrac{u}{v}$

Working: Using the poetry: $\dfrac{dy}{dx} = u'v^{-1} + -v^{-2}v'u$

Simplifying: $\dfrac{dy}{dx} = \dfrac{u'}{v} - \dfrac{v'u}{v^2}$

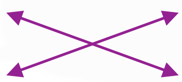
Common denominator: $\dfrac{dy}{dx} = \dfrac{u'v - v'u}{v^2}$ — this is the ***quotient rule***

E.g. 1 Find the derivative of these functions:

(a) $y = \dfrac{3x}{x^2+7}$ (b) $y = \dfrac{e^{6x}}{x^2}$ (c) $y = \dfrac{\sin 4x}{5x^3}$

Working: (a) $u = 3x$ $u' = 3$

$v = x^2 + 7$ $v' = 2x$



The arrows can be useful but you need to get the order right in the numerator as we are subtracting (Differentiate top, times by bottom)

$$\frac{dy}{dx} = \frac{3(x^2+7) - 2x \times 3x}{(x^2+7)^2}$$
$$\frac{dy}{dx} = \frac{3x^2 + 21 - 6x^2}{(x^2+7)^2}$$
$$\frac{dy}{dx} = \frac{3(7-x^2)}{(x^2+7)^2}$$

(b) $u = e^{6x}$ $\quad$ $u' = 6e^{6x}$

$v = x^2$ $\quad$ $v' = 2x$

$$\frac{dy}{dx} = \frac{6e^{6x} \times x^2 - 2x \times e^{6x}}{x^2}$$
$$\frac{dy}{dx} = \frac{xe^{6x}(3x-1)}{x^4}$$
$$\frac{dy}{dx} = \frac{2e^{6x}(3x-1)}{x^3}$$

(c) $u = \sin 4x$ $\quad$ $u' = 4\cos 4x$

$v = 5x^3$ $\quad$ $v' = 15x^2$

$$\frac{dy}{dx} = \frac{4\cos 4x \times 5x^3 - 15x^2 \times \sin 4x}{(5x^3)^2}$$
$$\frac{dy}{dx} = \frac{5x^2\left(4x\cos 4x - 3\sin 4x\right)}{25x^6}$$
$$\frac{dy}{dx} = \frac{4x\cos 4x - 3\sin 4x}{5x^4}$$

E.g. 2 Find the coordinates of the stationary point(s) of the curve $y = \dfrac{5x-4}{2x^2}$ and determine its/their nature.

Working:
$$\frac{dy}{dx} = \frac{5 \times 2x^2 - 4x \times (5x-4)}{(2x^2)^2}$$
$$\frac{dy}{dx} = \frac{-10x^2 + 16x}{4x^4}$$
$$\frac{dy}{dx} = \frac{8-5x}{2x^3}$$

A SP occurs when $\dfrac{dy}{dx} = 0$ so $\dfrac{8-5x}{2x^3} = 0$

Therefore, $8 - 5x = 0$
$$x = \frac{8}{5}$$

When $x = \dfrac{8}{5}$, $y = \dfrac{25}{32}$

Determining the nature

Gradient change method (easier than 2nd derivative method):

Choose an $x-$value either side of the stationary point and substitute into the first derivative.

When $x = 1$, $\dfrac{dy}{dx} = \dfrac{8 - 5 \times 1}{2 \times 1^3} > 0$

When $x = 2$, $\dfrac{dy}{dx} = \dfrac{8 - 5 \times 2}{2 \times 2^3} < 0$

The gradient goes +ve/0/−ve so we have a maximum at $\left(\dfrac{8}{5}, \dfrac{25}{32}\right)$

2nd derivative method $\dfrac{d^2y}{dx^2} = \dfrac{5x-12}{x^4}$

When $x = \dfrac{8}{5}$: $\dfrac{d^2y}{dx^2} = \dfrac{8-12}{\left(\frac{8}{5}\right)^4} < 0 \Rightarrow$ a maximum

Remember: $\sec x = \dfrac{1}{\cos x}$ $\quad cosec x = \dfrac{1}{\sin x}$ $\quad \cot x = \dfrac{1}{\tan x}$

E.g. 3 (a) By writing $y = \sec x$ as $y = \dfrac{1}{\cos x}$ and using the quotient rule, find $\dfrac{dy}{dx}$.

(b) Using a similar method, find $\dfrac{dy}{dx}$ when:

(i) $y = cosecx$

(ii) $y = \cot x$

Working: (a)

$$y = \frac{1}{\cos x} \Rightarrow$$
$$\frac{dy}{dx} = \frac{0 - -\sin x}{\cos x^2}$$
$$= \frac{\sin x}{\cos x^2}$$
$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

(b) (i)

$$y = \frac{1}{\sin x} \Rightarrow$$
$$\frac{dy}{dx} = \frac{0 - \cos x}{\sin x^2}$$
$$= -\frac{\cos x}{\sin x^2}$$
$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$
$$= -cosecx \cot x$$

(ii)

$$y = \frac{1}{\tan x} \Rightarrow$$
$$\frac{dy}{dx} = = \frac{0 - \sec^2 x}{\tan x^2}$$
$$= -\frac{\sec^2 x}{\tan x^2}$$
$$= -\frac{1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x}$$
$$= -\frac{1}{\sin x^2}$$
$$= -cosec^2 x$$

Video: Quotient rule

Differentiation methods EQ

Solutions to Starter and E.g.s

Exercise
p208 10C Qu 1i, 2-7