Recursive Sequences

Starter

1. (Review of GCSE material) Find the first 4 terms of the sequence given by $u_n = n(n+2)$.

Working: When $n = 1$, $u_1 = 1(1+2) = 3$
When $n = 2$, $u_2 = 2(2+2) = 8$
When $n = 3$, $u_3 = 3(3+2) = 15$
When $n = 4$, $u_4 = 4(4+2) = 24$

2. (Review of GCSE material) A sequence is generated by the recurrence relation $u_1 = 2$, $u_{n+1} = 3u_n + k$.
(a) Find u_3 in terms of k.
(b) Given that $u_4 = 28$, find k.

Working: (a) $u_1 = 2$
$u_{n+1} = 3u_n + k \quad \Rightarrow \quad u_2 = 3u_1 + k = 3 \times 2 + k = 6 + k$
$\therefore u_3 = 3u_2 + k = 3 \times (6 + k) + k = 18 + 4k$

(b) $u_4 = 3u_3 + k = 3 \times (18 + 4k) + k = 54 + 13k$
$u_4 = 28 \quad \Rightarrow \quad 54 + 13k = 28$
$13k = -26$
$k = -2$

E.g. 1 A sequence has $n-$term given by $u_n = 7n - 16$. Show that the sequence is increasing.

Working: $u_n = 7n - 16$,
$u_{n+1} = 7(n+1) - 16 = 7n - 9$
$u_{n+1} > u_{n+1} - 7 = 7n - 16 = u_n$.
Since $u_{n+1} > u_n$, the sequence in increasing

E.g. 2 A sequence is defined by the recurrence relation $u_1 = 100$, $u_{n+1} = 0.5u_n + 18$. Find the limit if it exists.

Working: Let L be the limit of the sequence.
Then $L = u_{n+1} = u_n$
$L = 0.5L + 18$
Solving gives $L = 36$

E.g. 3 In a sequence $u_1 = 6$, $u_2 = 7$ and $u_3 = 8.5$. If the recurrence relation is of the form $u_{n+1} = au_n + b$, find the values of the constants a and b.

Working: $u_{n+1} = au_n + b \quad \Rightarrow \quad u_2 = au_1 + b \quad \Rightarrow \quad 7 = 6a + b$
Similarly: $u_3 = au_2 + b \quad \Rightarrow \quad 8.5 = 7a + b$
Solving simultaneously gives $a = 1.5$, $b = -2$

E.g. 4 A linear recurrence relation is given by $u_{n+1} = au_n + b$. Given that the sequence converges find the limit value, L, in terms of a and b.

Working: If the sequence converges then $L = u_{n+1} = u_n$

So $L = aL + b \quad \Rightarrow \quad L(1-a) = b \quad \Rightarrow \quad L = \dfrac{b}{1-a}$

E.g. 5 A bank deposit account had a balance of £328 in August 2017, £504.40 in August 2018 and £689.62 in August 2019. The interest rate remained constant, no money was withdrawn but a capital amount was added each year. Calculate the interest rate and capital added per year.

Hint: Let the recurrence formula be $u_{n+1} = au_n + b$ where b is the capital added.

Working: $u_1 = 328$ and $u_{n+1} = au_n + b$

$u_2 = au_1 + b = 328a + b$ so $328a + b = 504.40$

$u_3 = au_2 + b = 504.40a + b$ so $504.40a + b = 689.62$

Solving simultaneously gives $a = 1.05$ and $b = 160$

So the interest rate is 5% and capital added is £160

E.g. 6* Two power companies share 100000 customers. PowerUs lose 20% of its customers to Light U each year. Light U loses 30% of its customers to PowerUs each year.

(a) Show that the recursive formula for PowerUs is $P_{n+1} = 0.8P_n + 0.3L_n$

(b) How many customers should each company have after a long period of time?

Hint: Two power companies share 100000 customers $\Rightarrow$ $P_n + L_n = 100000$

Working: (a) P loses 20% of own customers but gains 30% of L's customers

i.e. $P_{n+1} = 0.8P_n + 0.3L_n$

(b) The companies share 100000 customers so $P_n + L_n = 100000$

L loses 30% of own customers but gains 20% of P's customers

i.e. $L_{n+1} = 0.7L_n + 0.2P_n$

$P_{n+1} = 0.5P_n + 30000$

After a long period of time $P = P_{n+1} = P_n$ and $L = L_{n+1} = L_n$

$P = 0.8P + 0.3L$ so $2P = 3L$

Given that $P + L = 100000$ (total number of customers)

L = 60000 so PowerUs expects 60000 and LightU 40000 customers

Video: Recurrence relationships

Recurrence relationships EQ

Solutions to Starter and E.g.s

Exercise

p66 4A Qu 1-i, 2i, 3i, 4i, 5-10