Related Rates of Change

Starter

1. **(Review of last lesson)** A curve has parametric equations $x = 3t + 6$ and $y = 2t - 8$. Find the area between the curve, the $x-$axis and the points where $t = -2$ and $t = 2$.

Working: $\int_{-2}^{2} 3(2t-8)dt = -96$ so area $= 96$

2. Let $y = 3x^2$. The ***rate of change*** of y with respect to time, t, is 24. Use the ***chain rule*** to find the ***rate of change*** of x with respect to time when $x = 2$.

Working: "Find the ***rate of change*** of x with respect to time" means find $\frac{dx}{dt}$.

By the chain rule: $\frac{dx}{dt} = \frac{dy}{dt} \times \frac{dx}{dy} = \frac{\frac{dy}{dt}}{\frac{dy}{dx}}$

$\frac{dy}{dt} = 24$

From $y = 3x^2$ $\Rightarrow$ $\frac{dy}{dx} = 6x$ $\Rightarrow$ $\frac{dx}{dy} = \frac{1}{6x}$

$\therefore \frac{dx}{dt} = \frac{dy}{dt} \times \frac{dx}{dy} = 24 \times \frac{1}{6x} = \frac{4}{x}$

So when $x = 2$, $\frac{dx}{dt} = 2$.

3. A circle with area A is expanding at a rate of 5 cm² per second (5 $\text{cm}^2\ \text{s}^{-1}$). Find the exact rate of increase of the radius, r, of the circle when the radius is 4 cm.

Working: "Find the exact rate of increase of the radius" means find $\frac{dr}{dt}$.

A circle is expanding at a rate of 5 $\text{cm}^2\ \text{s}^{-1}$ $\Rightarrow$ $\frac{dA}{dt} = 5$

N.B. Area because cm^2

By the chain rule: $\frac{dr}{dt} = \frac{dA}{dt} \times \frac{dr}{dA} = \frac{\frac{dA}{dt}}{\frac{dA}{dr}}$ ***so we need*** $\frac{dA}{dr}$

For a circle $A = \pi r^2$ $\Rightarrow$ $\frac{dA}{dr} = 2\pi r$

$\therefore \frac{dr}{dt} = \frac{\frac{dA}{dt}}{\frac{dA}{dr}} = \frac{5}{2\pi r}$

So when $r = 4$, $\frac{dr}{dt} = \frac{5}{8\pi}$ cm s^{-1}

E.g. 1 Air is pumped into a spherical ball of volume V which expands at a rate of 8 cm^3 s^{-1}. Find the exact rate of increase of the radius, r, of the ball when the radius is 2 cm.

Working: "Find the exact rate of increase of the radius" means find $\frac{dr}{dt}$.

A spherical ball which expands at a rate of 8 cm^3 s^{-1} $\Rightarrow$ $\frac{dV}{dt} = 8$

By the chain rule: $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}}$ ***so we need*** $\frac{dV}{dr}$

For a sphere $V = \frac{4}{3}\pi r^3$ $\Rightarrow$ $\frac{dV}{dr} = 4\pi r^2$

$$\therefore \frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}} = \frac{8}{4\pi r^2} = \frac{2}{\pi r^2}$$

So when $r = 2$, $\frac{dr}{dt} = \frac{2}{\pi \times 2^2} = \frac{1}{2\pi}$ cm s^{-1}

E.g. 2 A cuboid has dimensions 6, x and x. The volume of the cuboid, V, is increasing at a rate of 4 cm^3 s^{-1}. Find the rate of increase of the side x when its length is 11 cm.

Working: "Find the rate of increase of the side x" means find $\frac{dx}{dt}$.

"…volume of the cuboid is increasing at a rate of 4 cm^3 s^{-1}" $\Rightarrow$ $\frac{dV}{dt} = 4$

By the chain rule: $\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dx}{dV} = \frac{\frac{dV}{dt}}{\frac{dV}{dx}}$ ***so we need*** $\frac{dV}{dx}$

For a cuboid, $V = 6x^2$ $\Rightarrow$ $\frac{dV}{dx} = 12x$

$$\therefore \frac{dx}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dx}} = \frac{4}{12x} = \frac{1}{3x}$$

So when $x = 11$, $\frac{dr}{dt} = \frac{1}{3 \times 11} = \frac{1}{33}$ cm s^{-1}

E.g. 3 The volume of a pyramid, V, is increasing at a rate of 16 m³/s. At this time, the area of the base, A, is 4 m² and it is increasing at a rate of 10 m²/s. Given also that the height, h, is 2 m, calculate the rate of change of height.

N.B. Volume of a pyramid $= \frac{1}{3} \times$ area of base $\times$ perpendicular height

Working: "V is increasing at a rate of 16 m³/s" $\Rightarrow \frac{dV}{dt} = 16$

"area of the base, A, is 4 m²" $\Rightarrow A = 4$

"it is increasing at a rate of 10 m²/s" $\Rightarrow \frac{dA}{dt} = 10$

Given also that the height, h, is 2 m $\Rightarrow h = 2$

From Volume of a pyramid $= \frac{1}{3} \times$ ***area of base*** $\times$ ***perpendicular height:***

$$V = \frac{1}{3}Ah$$

Differentiating implicitly with respect to t using the product rule:

$$\frac{dV}{dt} = \frac{1}{3}\frac{dA}{dt} \times h + \frac{dh}{dt} \times \frac{1}{3}A$$

$$\frac{dV}{dt} = \frac{1}{3}\left(h\frac{dA}{dt} + A\frac{dh}{dt}\right)$$

Substituting: $16 = \frac{1}{3}\left(2 \times 10 + 4\frac{dh}{dt}\right)$

Rearranging: $48 = 20 + 4\frac{dh}{dt}$

$$\frac{dh}{dt} = 7$$

Video: Related rates of change

Related rates of change EQ

Solutions to Starter and E.g.s

Exercise
p268 12E Qu 1i, 2i, 3i, 4-9