Repeated integration by parts

Starter

1. **(Review of last lesson)** Find $\int_0^{\frac{\pi}{3}} 18x \sin 3x \, dx$.

Working: Let $u = 18x \Rightarrow u' = 18$

Let $v' = \sin 3x \Rightarrow v = -\frac{1}{3}\cos 3x$

Using $\int uv' = uv - \int u'v$:

$$\int_0^{\frac{\pi}{3}} 18x \sin 3x \, dx = \left[18x \times -\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 18 \times \left(-\frac{1}{3}\cos 3x\right) dx$$
$$= \left[18x \times -\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} 6\cos 3x \, dx$$
$$= \Big[-6x\cos 3x + 2\sin 3x\Big]_0^{\frac{\pi}{3}}$$
$$= \left(-2\pi\cos\pi + 2\sin\pi\right) - \left(0 + 2\sin 0\right)$$
$$= 2\pi$$

2. Find $\int x^2 e^x \, dx$.

Working: Let $u = x^2 \Rightarrow u' = 2x$

Let $v' = e^x \Rightarrow v = e^x$

Using $\int uv' = uv - \int u'v$:

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

Since $\int 2xe^x \, dx$ cannot be done directly, integration by parts is used again.

Let $u = 2x \Rightarrow u' = 2$

Let $v' = e^x \Rightarrow v = e^x$

Using $\int uv' = uv - \int u'v$:

$$\int x^2 e^x \, dx = x^2 e^x - \left(2xe^x - \int 2e^x \, dx\right)$$
$$= x^2 e^x - 2xe^x + 2e^x + c$$
$$= e^x(x^2 - 2x + 2) + c$$

E.g. 1 Find: (a) $\int x^2 \sin x\,dx$ (b) $\int_0^1 x^2 e^{-2x}\,dx$

Working: (a) Let $u = x^2 \Rightarrow u' = 2x$

Let $v' = \sin x \Rightarrow v = -\cos x$

Using $\int uv' = uv - \int u'v$:

$$\int x^2 \sin x\,dx = -x^2\cos x - \int 2x \times (-\cos x)\,dx$$
$$= -x^2\cos x + \int 2x\cos x\,dx$$

Since $\int 2x\cos x\,dx$ cannot be done directly, integration by parts is used again.

Let $u = 2x \Rightarrow u' = 2$

Let $v' = \cos x \Rightarrow v = \sin x$

Using $\int uv' = uv - \int u'v$:

$$\int x^2 \sin x\,dx = -x^2\cos x + \left(2x\sin x - \int 2\sin x\,dx\right)$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

(b) Let $u = x^2 \Rightarrow u' = 2x$

Let $v' = e^{-2x} \Rightarrow v = -\frac{1}{2}e^{-2x}$

Using $\int uv' = uv - \int u'v$:

$$\int_0^1 x^2 e^{-2x}\,dx = \left[x^2 \times -\frac{1}{2}e^{-2x}\right]_0^1 - \int_0^1 2x \times -\frac{1}{2}e^{-2x}\,dx$$
$$= \left[-\frac{1}{2}x^2e^{-2x}\right]_0^1 + \int_0^1 xe^{-2x}\,dx$$

Since $\int xe^{-2x}\,dx$ cannot be done directly, integration by parts is used again.

Let $u = x \Rightarrow u' = 1$

Let $v' = e^{-2x} \Rightarrow v = -\frac{1}{2}e^{-2x}$

Using $\int uv' = uv - \int u'v$:

$$\int_0^1 x^2 e^{-2x}\,dx = \left[-\frac{1}{2}x^2e^{-2x}\right]_0^1 + \left[-\frac{1}{2}xe^{-2x}\right]_0^1 - \int_0^1 -\frac{1}{2}e^{-2x}\,dx$$
$$= \left[-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x}\right]_0^1 + \int_0^1 \frac{1}{2}e^{-2x}\,dx$$
$$= \left[-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1$$

$$= \left[\frac{1}{2}x^2e^{-2x} + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x}\right]_1^0$$
$$= \left(0 + 0 + \frac{1}{4}\right) - \left(\frac{1}{2}1^2e^{-2} + \frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}\right)$$
$$= \frac{1}{4} - \frac{5}{4e^2}$$

E.g. 2* Find $\int e^x \sin x\,dx$.

Working: Let $u = e^x \Rightarrow u' = e^x$
Let $v' = \sin x \Rightarrow v = -\cos x$

Using $\int uv' = uv - \int u'v$:

$$\int e^x \sin x\,dx = -e^x\cos x - \int e^x \times -\cos x\,dx$$
$$= -e^x\cos x + \int e^x\cos x\,dx$$

This doesn't seem to work since it is just as difficult to integrate $\int e^x \cos x\,dx$ as $\int e^x \sin x\,dx$ so let's try the other way around.

Let $u = \sin x \Rightarrow u' = \cos x$
Let $v' = e^x \Rightarrow v = e^x$

Using $\int uv' = uv - \int u'v$:

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

Back to square one…or is it.

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx$$
$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

By adding the equations, the integrals on the right-hand side cancel.

$$2\int e^x \sin x\,dx = e^x\sin x - e^x\cos x$$
$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c$$

Video: Repeated integration by parts

Solutions to Starter and E.g.s

Exercise
p234 11E Qu 1-3, 5-9