# Review of factor theorem

**Starter**

1. **(Review of previous material)**
Factorise the polynomial $x^3 + 2x^2 - x - 2$ as far as possible.

***Working:*** Let $f(x) = x^3 + 2x^2 - x - 2$
***Using the factor theorem:***
$f(1) = 1^3 + 2 \times 1^2 - 1 - 2 = 0 \quad \Rightarrow \quad x - 1$ is a factor
$x^3 + 2x^2 - x - 2 \equiv (x - 1)(ax^2 + bx + c)$
By inspection, $a = 1$ and $c = 2$
Equating coefficients of $x^2$: $\quad 2 = -a + b$
Since $a = 1$, $b = 3$
***N.B.*** $a$, $b$ and $c$ could have been found by polynomial division.
$x^3 + 2x^2 - x - 2 \equiv (x - 1)(x^2 + 3x + 2)$
***Now factorise the quadratic:***
$x^3 + 2x^2 - x - 2 \equiv (x - 1)(x + 1)(x + 2)$

***E.g. 1*** Decide whether the linear expression is a factor of the polynomial $f(x)$:
(a) $x - 3$ $\quad f(x) = 4x^3 - 4x^2 - 21x - 10$
(b) $2x + 1$ $\quad f(x) = 2x^3 + 3x^2 + 3x + 1$

***Working:*** (a) $f(3) = 4 \times 3^3 - 4 \times 3^2 - 21 \times 3 - 10 = -1 \neq 0$
Since $f(3) \neq 0$, $x - 3$ is not a factor.

(b) $$f\left(-\frac{1}{2}\right) = 2 \times \left(-\frac{1}{2}\right)^3 + 3 \times \left(-\frac{1}{2}\right)^2 + 3 \times \left(-\frac{1}{2}\right) + 1 = 0$$
Since $f\left(-\frac{1}{2}\right) = 0$, $2x + 1$ is a factor.

***E.g. 2*** Factorise the polynomial $x^3 + x^2 - 16x + 20$ as far as possible.

***Working:*** Let $f(x) = x^3 + x^2 - 16x + 20$
$f(2) = 2^3 + 2^2 - 16 \times 2 + 20 = 0$ so $x - 2$ is factor.
$x^3 + x^2 - 16x + 20 \equiv (x - 2)(ax^2 + bx + c)$
By inspection, $a = 1$ and $c = -10$
Equating coefficients of $x^2$: $\quad 1 = -2a + b$
Since $a = 1$, $b = 3$
***N.B.*** $a$, $b$ and $c$ could have been found by polynomial division.
$x^3 + x^2 - 16x + 20 \equiv (x - 2)(x^2 + 3x - 10)$
***Now factorise the quadratic:***
$x^3 + x^2 - 16x + 20 \equiv (x - 2)(x - 2)(x + 5)$

***E.g. 3*** The polynomial $9x^3 + 27x^2 - x - 3$ is the product of three linear factors. Find them.

***Working:***

Let $f(x) = 9x^3 + 27x^2 - x - 3$.
$f(-3) = 9 \times (-3)^3 + 27 \times (-3)^2 - (-3) - 3 = 0$ so $x + 3$ is factor.
$9x^3 + 27x^2 - x - 3 \equiv (x+3)(ax^2 + bx + c)$
By inspection, $a = 9$ and $c = -1$
Equating coefficients of $x^2$: $27 = 3a + 3b$
Since $a = 9$, $b = 0$
***N.B.*** $a$, $b$ and $c$ could have been found by polynomial division.
$9x^3 + 27x^2 - x - 3 \equiv (x+2)(9x^2 - 1)$
***Now factorise the quadratic:***
$x^3 + x^2 - 16x + 20 \equiv (x-2)(3x-1)(3x+1)$

***E.g. 4*** The polynomial $x^3 + ax^2 + bx - 6$ has factors $x + 1$ and $x - 2$. Find the values of $a$ and $b$.

***Working:***

Let $f(x) = x^3 + ax^2 + bx - 6$.
$x + 1$ is a factor $\Rightarrow f(-1) = 0$: $-1 + a - b - 6 = 0$
$a - b = 7$
$x - 2$ is a factor $\Rightarrow f(2) = 0$: $8 + 4a + 2b - 6 = 0$
$2a + b = -1$
Solving simultaneously gives: $a = 2$ and $b = -5$

***E.g. 5*** (a) Factorise $x^3 - 27$ into a linear and a quadratic factor.
(b) Hence or otherwise factorise: (i) $x^3 - y^3$ (ii) $x^3 + y^3$

***Working:***

(a) Let $f(x) = x^3 - 27$
$f(3) = 3^3 - 27 = 0$ so $x - 3$ is a factor.
$x^3 - 27 \equiv (x-3)(ax^2 + bx + c)$
By inspection, $a = 1$ and $c = 9$
Equating coefficients of $x^2$: $0 = -3a + b$
Since $a = 1$, $b = 3$
***N.B.*** $a$, $b$ and $c$ could have been found by polynomial division.
$x^3 - 27 \equiv (x-3)(x^2 + 3x + 9)$

(b) (i) $x^3 - 27 \equiv x^3 - 3^3$
So if: $x^3 - 27 \equiv (x-3)(x^2 + 3x + 9)$
then: $x^3 - 3^3 \equiv (x-3)(x^2 + 3x + 3^2)$
Replacing $3$ by $y$ gives:
$x^3 - y^3 \equiv (x-y)(x^2 + xy + y^2)$

(ii) $x^3 + y^3 = x^3 - (-y)^3$
Replace $y$ by $-y$ in the expression from (i)
$x^3 + y^3 \equiv (x - (-y))(x^2 + x \times (-y) + (-y)^2)$
$\equiv (x+y)(x^2 - xy + y^2)$

Video: Factor Theorem
Factor Theorem EQ

Solutions to Starter and E.g.s

**Exercise**
p95 5A Qu 1i, 2-7, (8-10 red)